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CERTIFIED REDUCED BASIS METHODS FOR PARAMETRIZED SADDLE POINT PROBLEMS

ANNA-LENA GERNER* AND KAREN VEROY†

Abstract. We present reduced basis approximations and associated rigorous *a posteriori* error bounds for parametrized saddle point problems. First, we develop new *a posteriori* error estimates that, unlike earlier approaches, provide upper bounds for the errors in the approximations of the primal variable and the Lagrange multiplier *separately*. The proposed method is an application of Brezzi's theory for saddle point problems to the reduced basis context, and exhibits significant advantages over existing methods. Second, based on an analysis of Brezzi's theory, we compare several options for the reduced basis approximation space from the perspective of approximation stability and computational cost. Finally, we introduce a new adaptive sampling procedure for saddle point problems constructing approximation spaces that are stable and, compared to earlier approaches, computationally much more efficient. The method is applied to a Stokes flow problem in a two-dimensional channel with a parametrized rectangular obstacle. Numerical results demonstrate: (i) the need to appropriately enrich the approximation space for the primal variable; (ii) the significant effects of different enrichment strategies; (iii) the rapid convergence of (stable) reduced basis approximations; and (iv) the advantages of the proposed error bounds with respect to sharpness and computational cost.

Key words. Saddle point problem; Stokes equations; incompressible fluid flow; model order reduction; reduced basis method; *a posteriori* error bounds; inf-sup condition

AMS subject classifications. 65N12, 65N15, 65N30, 76D07

Introduction. In many engineering applications, the aim is to optimize, control, or characterize in real-time a system whose behavior is governed by a partial differential equation (PDE). For the real-time or many-query context of PDE-constrained optimization, control, or characterization, classical discretization techniques such as finite element methods are generally too expensive. The reduced basis method, a model order reduction technique, is one means to address this difficulty.

The reduced basis (RB) method exploits the parametric structure of the governing PDE to construct rapidly convergent and computationally efficient approximations equipped with rigorous error bounds. The RB approximation is essentially a Galerkin projection onto a low-dimensional space that focuses on the solution manifold induced by the parametrized PDE. Rigorous *a posteriori* error bounds are derived as relaxations of the error-residual equation; in this paper, we make the crucial observation that RB *a posteriori* error bounds can be obtained through direct application of appropriate *a priori* stability estimates. Both RB approximations and error bounds are computed using an Offline–Online strategy enabling highly efficient (i.e., at minimal marginal cost) computations of the approximations and error bounds. Finally, RB approximations and error bounds are intimately linked through a greedy approach, in which the (Online-)inexpensive error bounds are used to construct the subsequent approximation spaces more optimally.

In this paper, we develop a new certified reduced basis method for parametrized saddle point problems. Saddle point problems often arise in practical applications where a certain quantity has to be minimized subject to a set of linear constraints;

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examples of such applications are mixed finite element methods (see, e.g., [5, 28]) or quadratic programming methods in optimal control (see, e.g., [19] and references therein). Although RB methods are well-developed for several classes of PDEs [13, 26, 31], parametrized saddle point problems pose additional difficulties that have not been fully addressed: Parameter-dependent constraints cause complications not only in the choice of *stable* RB approximation spaces [22, 29, 32], but also in the construction of rigorous and computationally efficient *a posteriori* error bounds. Saddle point systems with parameter-dependent constraints occur, for example, when applying the RB method to incompressible fluid flow in parametrized domains. Earlier work on the Stokes and incompressible Navier–Stokes equations has established the method for non-parametrized domains [17, 36], and there have been several efforts [8, 20, 23, 27, 29, 32] at extending the method to parametrized domains. However, in these earlier examples, either rigorous error bounds were not treated; or the geometric variations considered are very small or applicable only to a very limited set of problems. A penalty approach presented in [10] provides very efficient error bounds admitting geometric variations with relative ease, but at the expense of an additional error in the finite element "truth" approximation upon which the RB approximation is built. Hence, there is still a lot of room for improvement.

In this work, we shall focus on two important aspects. First, we present new *a posteriori* error estimates that, unlike earlier approaches, provide upper bounds for the errors in the approximations for u and p *separately* (see [39] for initial investigations). The proposed method is a direct application of Brezzi's theory for saddle point problems [4, 5] to the reduced basis context, and exhibits significant advantages over existing *a posteriori* error estimators based on either Babuška's theory for noncoercive problems [24, 37] or a penalty approach [10]. Second, in view of approximation stability and computational cost, we shall analyze and compare several options for the RB approximation spaces. Through numerical tests, we illustrate the significant savings (compared to earlier strategies [27, 32]) achievable by enriching the RB approximation space for the primal variable appropriately. Finally, both *a posteriori* error bounds and enrichment strategies are employed in an adaptive sampling procedure for constructing RB approximation spaces that are not only stable but also efficient.

The paper is organized as follows. In Section 1, we state the general problem setting. We introduce the variational form of a parametrized saddle point problem, define the "truth" approximation upon which we shall build our RB approximation, and briefly illustrate a Stokes flow model problem for which we shall present numerical results. In Section 2, we describe our RB method for saddle point problems. In §2.1, we define the RB approximation as the Galerkin projection onto a low-dimensional RB approximation space. We develop rigorous *a posteriori* error bounds in §2.2 and discuss how to construct the RB approximation space in §2.3; computational efficiency is achieved by the RB Offline–Online strategy summarized in §2.4. In §2.5, a new adaptive sampling procedure combines both error bounds derived in §2.2 and observations in §2.3 to identify RB approximation spaces that are stable and computationally efficient. Finally, in Section 3, we give some concluding remarks.

1. Parametrized Saddle Point Problems.

1.1. General Problem Statement. Let $X_{\rm e}$ and $Y_{\rm e}$ be two Hilbert spaces defined over the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$, with inner products $(\cdot,\cdot)_{X_{\rm e}}$, $(\cdot,\cdot)_{Y_{\rm e}}$ and associated norms $\|\cdot\|_{X_{\rm e}} = \sqrt{(\cdot,\cdot)_{X_{\rm e}}}$, $\|\cdot\|_{Y_{\rm e}} = \sqrt{(\cdot,\cdot)_{Y_{\rm e}}}$, respectively.[1] We define the combined space $Z_{\rm e} \equiv X_{\rm e} \times Y_{\rm e}$, with inner product $(\cdot,\cdot)_{Z_{\rm e}} \equiv (\cdot,\cdot)_{X_{\rm e}} + (\cdot,\cdot)_{Y_{\rm e}}$ and norm $\|\cdot\|_{Z_{\rm e}} = \sqrt{(\cdot,\cdot)_{Z_{\rm e}}}$. The associated dual spaces are denoted by $X'_{\rm e}$, $Y'_{\rm e}$, and $Z'_{\rm e}$.

Furthermore, let $\mathcal{D} \subset \mathbb{R}^n$ be a prescribed n-dimensional, compact parameter set. For any parameter $\mu \in \mathcal{D}$, $a(\cdot,\cdot;\mu) : X_{\rm e} \times X_{\rm e} \to \mathbb{R}$ and $b(\cdot,\cdot;\mu) : X_{\rm e} \times Y_{\rm e} \to \mathbb{R}$ then denote continuous bilinear forms,

$$\gamma_a^{\rm e}(\mu) \equiv \sup_{u\in X_{\rm e}} \sup_{v\in X_{\rm e}} \frac{a(u,v;\mu)}{\|u\|_{X_{\rm e}}\|v\|_{X_{\rm e}}} < \infty, \quad \forall\, \mu \in \mathcal{D}, \tag{1.1}$$

$$\gamma_b^{\rm e}(\mu) \equiv \sup_{q\in Y_{\rm e}} \sup_{v\in X_{\rm e}} \frac{b(v,q;\mu)}{\|q\|_{Y_{\rm e}}\|v\|_{X_{\rm e}}} < \infty, \quad \forall\, \mu \in \mathcal{D}. \tag{1.2}$$

We moreover assume that $a(\cdot,\cdot;\mu)$ is coercive on $X_{\rm e}$,

$$\alpha_a^{\rm e}(\mu) \equiv \inf_{v\in X_{\rm e}} \frac{a(v,v;\mu)}{\|v\|^2_{X_{\rm e}}} > 0, \quad \forall\, \mu \in \mathcal{D}, \tag{1.3}$$

and that $b(\cdot,\cdot;\mu)$ satisfies the inf-sup condition

$$\beta_{\rm Br}^{\rm e}(\mu) \equiv \inf_{q\in Y_{\rm e}} \sup_{v\in X_{\rm e}} \frac{b(v,q;\mu)}{\|q\|_{Y_{\rm e}}\|v\|_{X_{\rm e}}} > 0, \quad \forall\, \mu \in \mathcal{D}. \tag{1.4}$$

We shall refer to $\beta_{\rm Br}^{\rm e}(\mu)$ as the "exact" Brezzi inf-sup constant. We additionally define the "exact" Babuška inf-sup constant

$$\beta_{\rm Ba}^{\rm e}(\mu) \equiv \inf_{(u,p)\in Z_{\rm e}} \sup_{(v,q)\in Z_{\rm e}} \frac{a(u,v;\mu) + b(v,p;\mu) + b(u,q;\mu)}{\|(u,p)\|_{Z_{\rm e}}\|(v,q)\|_{Z_{\rm e}}}, \quad \forall\, \mu \in \mathcal{D}; \tag{1.5}$$

it follows that $\beta_{\rm Ba}^{\rm e}(\mu) > 0$ for all $\mu \in \mathcal{D}$ from the assumptions (1.1)–(1.4) (see [1, 4]).

We now consider the following variational formulation of a general parametrized saddle point problem: For any $\mu \in \mathcal{D}$, find $(u_{\rm e}(\mu), p_{\rm e}(\mu)) \in X_{\rm e} \times Y_{\rm e}$ such that

$$\begin{aligned} a(u_{\rm e}(\mu),v;\mu) + b(v,p_{\rm e}(\mu);\mu) &= f(v;\mu), \quad \forall\, v \in X_{\rm e},\\ b(u_{\rm e}(\mu),q;\mu) &= g(q;\mu), \quad \forall\, q \in Y_{\rm e}, \end{aligned} \tag{1.6}$$

where $f(\cdot;\mu)$ and $g(\cdot;\mu)$ are bounded linear functionals in $X'_{\rm e}$ and $Y'_{\rm e}$, respectively. From the results of Brezzi [4] (see, e.g., also [5, 9, 11]), it is well-known that under the assumptions (1.1), (1.2), (1.3), and (1.4), the above saddle point problem (1.6) is well-posed and has a unique solution for any $f(\cdot;\mu) \in X'_{\rm e}$, $g(\cdot;\mu) \in Y'_{\rm e}$.

We pause at this point to briefly comment on the nature of the parameter dependence of our linear and bilinear forms. The efficiency of the reduced basis method relies on an Offline–Online computational decomposition strategy that requires that all bilinear and linear forms in (1.6) depend affinely on the parameter μ. For instance, we assume that for some $Q_a \in \mathbb{N}$, the bilinear form $a(\cdot,\cdot;\mu)$ can be written as

$$a(u,v;\mu) = \sum_{k=1}^{Q_a} \Theta_a^k(\mu)\, a^k(u,v), \quad \forall\, u,v \in X_{\rm e}, \tag{1.7}$$

where for $1 \le k \le Q_a$, the parameter-dependent coefficient functions $\Theta_a^k(\mu)$ are continuous over the parameter set $\mathcal{D}$; the parameter-independent bilinear forms $a^k(\cdot,\cdot)$ are continuous on $X_{\rm e} \times X_{\rm e}$. We assume analogous representations for the bilinear form $b(\cdot,\cdot;\mu)$ and the linear functionals $f(\cdot;\mu)$ and $g(\cdot;\mu)$.

[1] Here and in the following, the subscript e denotes "exact".

1.2. Truth Approximation. We now introduce a high-fidelity "truth" finite element approximation upon which our RB approximation will subsequently be built. Let $\mathcal{T}_\Omega$ be a regular triangulation of Ω, and let X and Y be conforming finite element approximation subspaces of X_e and Y_e over $\mathcal{T}_\Omega$. We then define the combined space $Z \equiv X \times Y$, and denote by $\mathcal{N}$ the dimension of Z. We emphasize that X, Y, and Z are *finite*-dimensional, and that the dimension $\mathcal{N}$ is typically very large. Finally, the "truth" approximation subspaces inherit the inner products and norms of the exact spaces: $(\cdot,\cdot)_X \equiv (\cdot,\cdot)_{X_\mathrm{e}}$, $\|\cdot\|_X \equiv \|\cdot\|_{X_\mathrm{e}}$, $(\cdot,\cdot)_Y \equiv (\cdot,\cdot)_{Y_\mathrm{e}}$, $\|\cdot\|_Y \equiv \|\cdot\|_{Y_\mathrm{e}}$, and $(\cdot,\cdot)_Z \equiv (\cdot,\cdot)_{Z_\mathrm{e}}$, $\|\cdot\|_Z \equiv \|\cdot\|_{Z_\mathrm{e}}$.

The continuity and coercivity properties (1.1), (1.2), and (1.3) are clearly inherited by the "truth" approximation spaces,

$$\gamma_a(\mu) \equiv \sup_{u\in X} \sup_{v\in X} \frac{a(u,v;\mu)}{\|u\|_X \|v\|_X} < \infty, \tag{1.8}$$

$$\gamma_b(\mu) \equiv \sup_{q\in Y} \sup_{v\in X} \frac{b(v,q;\mu)}{\|q\|_Y \|v\|_X} < \infty, \tag{1.9}$$

$$\alpha_a(\mu) \equiv \inf_{v\in X} \frac{a(v,v;\mu)}{\|v\|_X^2} > 0, \tag{1.10}$$

for all $\mu \in \mathcal{D}$. We now further assume that the approximation spaces X and Y are chosen such that they satisfy the Ladyzhenskaya-Babuška-Brezzi (LBB) inf-sup condition (see, e.g., [5])

$$\beta_\mathrm{Br}(\mu) \equiv \inf_{q\in Y} \sup_{v\in X} \frac{b(v,q;\mu)}{\|q\|_Y \|v\|_X} \geq \beta^0_\mathrm{Br}(\mu) > 0, \quad \forall\, \mu \in \mathcal{D}, \tag{1.11}$$

where $\beta^0_\mathrm{Br}(\mu)$ is a constant independent of the dimension $\mathcal{N}$. We refer to $\beta_\mathrm{Br}(\mu)$ as the "truth" Brezzi inf-sup constant. We shall also require the "truth" Babuška inf-sup constant

$$\beta_\mathrm{Ba}(\mu) \equiv \inf_{(u,p)\in Z} \sup_{(v,q)\in Z} \frac{a(u,v;\mu) + b(v,p;\mu) + b(u,q;\mu)}{\|(u,p)\|_Z \|(v,q)\|_Z}, \quad \forall\, \mu \in \mathcal{D}; \tag{1.12}$$

again, it follows from (1.8)–(1.11) that $\beta_\mathrm{Ba}(\mu) > 0$ for all $\mu \in \mathcal{D}$. In particular, $\beta_\mathrm{Br}(\mu) \geq \beta_\mathrm{Ba}(\mu) \geq C(\alpha_a(\mu), \gamma_a(\mu), \beta_\mathrm{Br}(\mu)) > 0$ for all $\mu \in \mathcal{D}$; see, e.g., [38] for an explicit representation of $C(\alpha_a(\mu), \gamma_a(\mu), \beta_\mathrm{Br}(\mu))$.

We now define our "truth" finite element approximations to be the Galerkin projections of $u_\mathrm{e}(\mu) \in X_\mathrm{e}$ and $p_\mathrm{e}(\mu) \in Y_\mathrm{e}$ onto X and Y, respectively: Given $\mu \in \mathcal{D}$, we find $(u(\mu), p(\mu)) \in X \times Y$ such that

$$\begin{aligned} a(u(\mu),v;\mu) + b(v,p(\mu);\mu) &= f(v;\mu), \quad \forall\, v \in X, \\ b(u(\mu),q;\mu) &= g(q;\mu), \quad \forall\, q \in Y. \end{aligned} \tag{1.13}$$

As for the exact problem in §1.1, it follows from (1.8), (1.9), (1.10), and (1.11) that for all $\mu \in \mathcal{D}$, the "truth" model problem (1.13) has a unique solution for any $f(\cdot;\mu) \in X'_\mathrm{e}$, $g(\cdot;\mu) \in Y'_\mathrm{e}$. (For more details on finite element approximations for saddle point problems, see, e.g., [5, 9, 11].)

1.3. Incompressible Fluid Flow in Parametrized Domains. Motivated by applications in microfluidics [34], we present numerical results for a Stokes flow in a two-dimensional channel with a rectangular obstacle as illustrated as in Fig. 1.1.

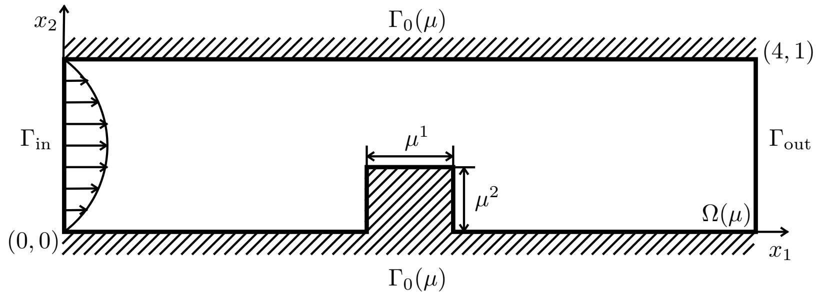


FIG. 1.1. *Physical domain: a parametrized two-dimensional microchannel with an obstacle. The parameters of interest are width μ^1 and height μ^2 of the obstacle.*

The problem depends on two geometric parameters $\mu \equiv (\mu^1, \mu^2)$ in $\mathcal{D} \equiv [0.1, 0.5]^2$, representing width μ^1 and height μ^2 of the obstacle $\mathcal{O}(\mu) \equiv [2 - \frac{\mu^1}{2}, 2 + \frac{\mu^1}{2}] \times [0, \mu^2]$. We denote the physical domain by $\Omega(\mu) \equiv \big((0,4) \times (0,1)\big) \setminus \mathcal{O}(\mu)$ with its boundary $\Gamma(\mu)$. We assume fully-developed flow conditions with a parabolic velocity profile on the inflow boundary $\Gamma_{\rm in} \equiv \{0\} \times [0,1]$, natural outflow conditions on $\Gamma_{\rm out} \equiv \{4\} \times [0,1]$, and no-penetration and no-slip velocity conditions on the channel walls and obstacle boundary $\Gamma_0(\mu) \equiv \Gamma(\mu) \setminus (\Gamma_{\rm in} \cup \Gamma_{\rm out})$. The problem statement over the physical domain $\Omega(\mu)$ is then transformed to an equivalent problem posed over a *parameter-independent* reference domain Ω. For this purpose, $\Omega(\mu)$ is traced back to $\Omega \equiv \Omega(\mu_{\rm ref})$ by a continuous, piecewise affine mapping; we denote the boundary of Ω by $\Gamma \equiv \Gamma(\mu_{\rm ref})$, $\Gamma_0 \equiv \Gamma_0(\mu_{\rm ref})$. For further details on the mapping procedure for this particular model problem, we refer to [10]; for more general problems, we refer to [31]. (See [20] and references therein for a different approach based on the Piola transformation.)

Our exact function spaces in §1.1 are now given by $X_{\rm e} \equiv \{ v \in (H^1(\Omega))^2 \mid v|_{\Gamma_0 \cup \Gamma_{\rm in}} = 0 \}$, $Y_{\rm e} \equiv L^2(\Omega)$, where $H^1(\Omega) \equiv \{ v \in L^2(\Omega) \mid \nabla v \in (L^2(\Omega))^2 \}$, and $L^2(\Omega)$ is the space of square integrable functions over Ω. On these spaces, we consider the inner products and associated norms

$$(v, w)_{X_{\rm e}} \equiv \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j}, \quad \forall\, v, w \in X_{\rm e}, \quad \|\cdot\|_{X_{\rm e}} = \sqrt{(\cdot,\cdot)_{X_{\rm e}}},$$

$$(p, q)_{Y_{\rm e}} \equiv \int_\Omega p\, q, \quad \forall\, p, q \in Y_{\rm e}, \quad \|\cdot\|_{Y_{\rm e}} = \sqrt{(\cdot,\cdot)_{Y_{\rm e}}}.$$

The bilinear forms $a(\cdot,\cdot;\mu)$ and $b(\cdot,\cdot;\mu)$ in the variational formulation (1.6) for the Stokes equations satisfy all the assumptions (1.1), (1.2), (1.3), (1.4), and (1.7) introduced in §1.1 (see, e.g., [5]). The exact representation of the bilinear and linear forms for this model problem is given in [10]; we particularly have $Q_a = 10$, $Q_b = 6$, and $Q_f = Q_g = 1$ in the respective μ-affine expansions (1.7). We then choose the truth approximation spaces X and Y in §1.2 as the standard conforming $\mathbb{P}_2$–$\mathbb{P}_1$ (quadratic–linear) Taylor–Hood finite element approximation subspaces [35] over the regular triangulation $\mathcal{T}_\Omega$. It is a well-known result (see, e.g., [5, 11, 28]) that in this case, X and Y indeed satisfy the LBB inf-sup condition (1.11). Based on a fine mesh with 16,602 elements, the truth system (1.13) has a dimension of $\mathcal{N} = 72{,}076$.

In this paper, all numerical results are attained using the open source software `rbOOmit` [18], an implementation of the RB framework within the C++ finite element library `libMesh` [16].

2. The Reduced Basis Method. We now turn to the reduced basis (RB) method, discussing the approximation procedure, rigorous *a posteriori* error estimators, and the construction of *stable* approximation spaces.

2.1. Galerkin Projection. Let us suppose for now that we are given a set of nested, low-dimensional reduced basis approximation subspaces $X_N \subset X_{N+1} \subset X$ and $Y_N \subset Y_{N+1} \subset Y$, $N \in \mathbb{N}_{\max} \equiv \{1, \ldots, N_{\max}\}$. We denote by N_X and N_Y the dimensions of X_N and Y_N, respectively, and the total dimension of $Z_N \equiv X_N \times Y_N$ by $N_Z \equiv N_X + N_Y$. The subspaces X_N, Y_N, and Z_N again inherit all inner products and norms of X, Y, and Z, respectively. The RB approximation is then defined as the Galerkin projection onto these low-dimensional subspaces: For $\mu \in \mathcal{D}$, we find $u_N(\mu) \in X_N$ and $p_N(\mu) \in Y_N$ such that

$$\begin{aligned} a(u_N(\mu), v_N; \mu) + b(v_N, p_N(\mu); \mu) &= f(v_N; \mu), \quad \forall\, v_N \in X_N, \\ b(u_N(\mu), q_N; \mu) &= g(q_N; \mu), \quad \forall\, q_N \in Y_N. \end{aligned} \tag{2.1}$$

2.2. Rigorous *A Posteriori* Error Estimation. We now aim to develop not only efficient reduced order approximations, but also *a posteriori* error estimates that are rigorous, sharp, and computationally inexpensive. In this section, we assume that the low-dimensional RB approximation spaces X_N, Y_N are constructed such that for any given parameter $\mu \in \mathcal{D}$, a solution $(u_N(\mu), p_N(\mu)) \in X_N \times Y_N$ to (2.1) exists. We then denote the errors in the RB approximations $u_N(\mu) \in X_N$, $p_N(\mu) \in Y_N$, and $(u_N(\mu), p_N(\mu)) \in Z_N$ with respect to the truth approximations by

$$\begin{aligned} e_N^u(\mu) &\equiv u(\mu) - u_N(\mu) \in X, \\ e_N^p(\mu) &\equiv p(\mu) - p_N(\mu) \in Y, \\ e_N(\mu) &\equiv (e_N^u(\mu), e_N^p(\mu)) \in Z. \end{aligned} \tag{2.2}$$

To formulate rigorous and computationally inexpensive upper bounds for the respective errors, we first have to introduce further ingredients. The first set of ingredients consists of computationally (Online-)efficient lower and upper bounds to the truth continuity and coercivity constants (1.8) and (1.10),

$$\begin{aligned} \gamma_a^{\mathrm{LB}}(\mu) \;\le\; \gamma_a(\mu) \;\le\; \gamma_a^{\mathrm{UB}}(\mu), \\ \alpha_a^{\mathrm{LB}}(\mu) \;\le\; \alpha_a(\mu) \;\le\; \alpha_a^{\mathrm{UB}}(\mu), \end{aligned} \quad \forall\, \mu \in \mathcal{D}, \tag{2.3}$$

and to the truth Brezzi and Babuška inf-sup constants (1.11) and (1.12),

$$\begin{aligned} \beta_{\mathrm{Br}}^{\mathrm{LB}}(\mu) \;\le\; \beta_{\mathrm{Br}}(\mu) \;\le\; \beta_{\mathrm{Br}}^{\mathrm{UB}}(\mu), \\ \beta_{\mathrm{Ba}}^{\mathrm{LB}}(\mu) \;\le\; \beta_{\mathrm{Ba}}(\mu) \;\le\; \beta_{\mathrm{Ba}}^{\mathrm{UB}}(\mu), \end{aligned} \quad \forall\, \mu \in \mathcal{D}. \tag{2.4}$$

The second set of ingredients consists of dual norms of the residuals associated with the RB approximation,

$$\|r_N^1(\cdot;\mu)\|_{X'} = \sup_{v \in X} \frac{r_N^1(v;\mu)}{\|v\|_X}, \quad \|r_N^2(\cdot;\mu)\|_{Y'} = \sup_{q \in Y} \frac{r_N^2(q;\mu)}{\|q\|_Y}, \tag{2.5}$$

where, for all $\mu \in \mathcal{D}$, $r_N^1(\cdot;\mu) \in X'$ and $r_N^2(\cdot;\mu) \in Y'$ are defined as

$$r_N^1(v;\mu) \equiv f(v;\mu) - a(u_N(\mu), v;\mu) - b(v, p_N(\mu);\mu), \quad \forall\, v \in X, \tag{2.6}$$

$$r_N^2(q;\mu) \equiv g(q;\mu) - b(u_N(\mu), q;\mu), \quad \forall\, q \in Y. \tag{2.7}$$

For all $\mu \in \mathcal{D}$, the total residual is then given by $r_N((v,q);\mu) \equiv r_N^1(v;\mu) + r_N^2(q;\mu)$, $\forall\, (v,q) \in Z$, with its dual norm

$$\|r_N(\cdot;\mu)\|_{Z'} = \sup_{(v,q)\in Z} \frac{r_N((v,q);\mu)}{\|(v,q)\|_Z} = \sqrt{\|r_N^1(\cdot;\mu)\|_{X'}^2 + \|r_N^2(\cdot;\mu)\|_{Y'}^2}. \tag{2.8}$$

We now formulate *a posteriori* error bounds for the respective errors (2.2) in the RB approximation.

Corollary 2.1. *For any given $\mu \in \mathcal{D}$, $N \in \mathbb{N}_{\max}$, and $\beta_{\mathrm{Ba}}^{\mathrm{LB}}(\mu)$ satisfying (2.4), we define*

$$\Delta_N^{\mathrm{Ba}}(\mu) \equiv \frac{\|r_N(\cdot;\mu)\|_{Z'}}{\beta_{\mathrm{Ba}}^{\mathrm{LB}}(\mu)}. \tag{2.9}$$

Then, $\Delta_N^{\mathrm{Ba}}(\mu)$ is an upper bound for the error $e_N(\mu)$,

$$\|e_N(\mu)\|_Z \le \Delta_N^{\mathrm{Ba}}(\mu), \quad \forall\, \mu \in \mathcal{D},\ \forall\, N \in \mathbb{N}_{\max}. \tag{2.10}$$

The above error bound directly follows from the Banach-Nečas-Babuška Theorem [1, 9] and represents a well-known result for noncoercive problems [8, 23, 36, 37].

Since many engineering outputs of interest such as average vorticity or pressure drop depend on either the primal variable (velocity) $u(\mu)$ or the Lagrange multiplier (pressure) $p(\mu)$, we aim to develop *separate* bounds for the errors in the RB approximations $u_N(\mu)$ and $p_N(\mu)$. Noting that *a posteriori* RB error bounds for coercive and general noncoercive problems can be derived from standard *a priori* stability results, we apply Brezzi's theory for saddle point problems [4] to our problem setting. Since some aspects of this theory are important also to the construction of RB approximation spaces in §2.3 and §2.5, we summarize the proof below.

Corollary 2.2. *For any given $\mu \in \mathcal{D}$, $N \in \mathbb{N}_{\max}$, and $\alpha_a^{\mathrm{LB}}(\mu)$, $\gamma_a^{\mathrm{UB}}(\mu)$, $\beta_{\mathrm{Br}}^{\mathrm{LB}}(\mu)$ satisfying (2.3) and (2.4), we define*

$$\Delta_N^u(\mu) \equiv \frac{\|r_N^1(\cdot;\mu)\|_{X'}}{\alpha_a^{\mathrm{LB}}(\mu)} + \left(1 + \frac{\gamma_a^{\mathrm{UB}}(\mu)}{\alpha_a^{\mathrm{LB}}(\mu)}\right) \frac{\|r_N^2(\cdot;\mu)\|_{Y'}}{\beta_{\mathrm{Br}}^{\mathrm{LB}}(\mu)}, \tag{2.11}$$

$$\Delta_N^p(\mu) \equiv \frac{\|r_N^1(\cdot;\mu)\|_{X'}}{\beta_{\mathrm{Br}}^{\mathrm{LB}}(\mu)} + \frac{\gamma_a^{\mathrm{UB}}(\mu)}{\beta_{\mathrm{Br}}^{\mathrm{LB}}(\mu)} \Delta_N^u(\mu). \tag{2.12}$$

Then, $\Delta_N^u(\mu)$ and $\Delta_N^p(\mu)$ are upper bounds for the errors $e_N^u(\mu)$ and $e_N^p(\mu)$,

$$\|e_N^u(\mu)\|_X \le \Delta_N^u(\mu), \quad \|e_N^p(\mu)\|_Y \le \Delta_N^p(\mu), \quad \forall\, \mu \in \mathcal{D},\ \forall\, N \in \mathbb{N}_{\max}. \tag{2.13}$$

Proof. Let $\mu \in \mathcal{D}$ and $N \in \mathbb{N}_{\max}$. By (2.2), (2.6), (2.7), and (1.13), the errors $e_N^u(\mu) \in X$ and $e_N^p(\mu) \in Y$ satisfy the equations

$$a(e_N^u(\mu), v; \mu) + b(v, e_N^p(\mu); \mu) = r_N^1(v;\mu), \quad \forall\, v \in X, \tag{2.14}$$

$$b(e_N^u(\mu), q; \mu) = r_N^2(q;\mu), \quad \forall\, q \in Y. \tag{2.15}$$

The bilinear forms $a(\cdot,\cdot;\mu)$ and $b(\cdot,\cdot;\mu)$ define bounded linear operators $A(\mu) : X \to X'$, $B(\mu) : X \to Y'$ and its transpose $B(\mu)^t : Y \to X'$ by

$$\langle A(\mu)\, u, v\rangle = a(u, v; \mu), \qquad \forall\, u, v \in X,$$
$$\langle B(\mu)\, v, q\rangle = b(v, q; \mu) = \langle B(\mu)^t q, v\rangle, \quad \forall\, v \in X,\ q \in Y;$$

here, $\langle\cdot,\cdot\rangle$ denotes the respective dual pairing. The error in the approximation of the primal variable may now be uniquely decomposed into $e_N^u(\mu) = e_N^0(\mu) + e_N^\perp(\mu)$, where $e_N^0(\mu) \in \ker(B(\mu))$ and $e_N^\perp(\mu) \in \ker(B(\mu))^\perp \equiv X/\ker(B(\mu))$. From (2.14), $e_N^0(\mu) \in \ker(B(\mu))$ then solves

$$a(e_N^0(\mu), v; \mu) = r_N^1(v; \mu) - a(e_N^\perp(\mu), v; \mu), \quad \forall\, v \in \ker(B(\mu)),$$

and it is thus bounded by

$$\|e_N^0(\mu)\|_X \le \frac{1}{\alpha_a(\mu)} \sup_{v\in\ker(B(\mu))} \frac{r_N^1(v;\mu) - a(e_N^\perp(\mu), v; \mu)}{\|v\|_X} \tag{2.16}$$

$$\le \frac{1}{\alpha_a(\mu)}\Big(\|r_N^1(\cdot;\mu)\|_{X'} + \gamma_a(\mu)\|e_N^\perp(\mu)\|_X\Big), \tag{2.17}$$

with (1.8), (1.10), and the classical Lax-Milgram Lemma (see, e.g., [9, 28]). By the LBB inf-sup condition (1.11), there particularly also holds (see [4], [5, §II.1, Proposition 1.2])

$$\beta_{\mathrm{Br}}(\mu) = \inf_{v\in\ker(B(\mu))^\perp} \sup_{q\in Y} \frac{b(v,q;\mu)}{\|v\|_X\|q\|_Y}.$$

Applied to $e_N^\perp(\mu) \in \ker(B(\mu))^\perp$, this yields

$$\|e_N^\perp(\mu)\|_X \le \frac{1}{\beta_{\mathrm{Br}}(\mu)} \sup_{q\in Y} \frac{b(e_N^\perp(\mu), q; \mu)}{\|q\|_Y} = \frac{\|r_N^2(\cdot;\mu)\|_{Y'}}{\beta_{\mathrm{Br}}(\mu)}, \tag{2.18}$$

where the equality follows from $B(\mu)\, e_N^\perp(\mu) = B(\mu)\, e_N^u(\mu)$ in Y' with (2.15). We then obtain the bound (2.11) for $\|e_N^u(\mu)\|_X$ by combining (2.17), (2.18), and (2.3), (2.4).

We now turn to the error in the approximation of the Lagrange multiplier. From (1.11) and (2.14), we derive

$$\|e_N^p(\mu)\|_Y \le \frac{1}{\beta_{\mathrm{Br}}(\mu)} \sup_{v\in X} \frac{b(v, e_N^p(\mu);\mu)}{\|v\|_X} = \frac{1}{\beta_{\mathrm{Br}}(\mu)} \sup_{v\in X} \frac{r_N^1(v;\mu) - a(e_N^u(\mu), v;\mu)}{\|v\|_X}$$

$$\le \frac{1}{\beta_{\mathrm{Br}}(\mu)}\Big(\|r_N^1(\cdot;\mu)\|_{X'} + \gamma_a(\mu)\,\|e_N^u(\mu)\|_X\Big). \tag{2.19}$$

Together with (2.11) and again (2.3), (2.4), this finally yields (2.12) and (2.13). ◻

Clearly, we can construct another rigorous bound $\Delta_N^{\mathrm{Br}}(\mu)$ for the combined error $e_N(\mu) = (e_N^u(\mu), e_N^p(\mu))$: From (2.13), we obtain

$$\|e_N(\mu)\|_Z \le \sqrt{\big(\Delta_N^u(\mu)\big)^2 + \big(\Delta_N^p(\mu)\big)^2} \equiv \Delta_N^{\mathrm{Br}}(\mu), \quad \forall\, \mu \in \mathcal{D},\ N \in \mathbb{N}_{\max}. \tag{2.20}$$

We shall not only examine the above *a posteriori* error bounds with respect to sharpness and computational efficiency (see §2.4 and §2.6), but also emphasize their importance to the construction of efficient RB approximation spaces (see §2.5).

REMARK 2.3. We here add some comments on the sharpness of the above *a posteriori* error bounds. Again, let μ be any parameter in $\mathcal{D}$ and $N \in \mathbb{N}_{\max}$.

First, in addition to (2.13), it is clearly also true that

$$\begin{aligned} \|r_N^1(\cdot;\mu)\|_{X'} &\le \gamma_a(\mu)\|e_N^u(\mu)\|_X + \gamma_b(\mu)\|e_N^p(\mu)\|_Y, \\ \|r_N^2(\cdot;\mu)\|_{Y'} &\le \gamma_b(\mu)\|e_N^u(\mu)\|_X. \end{aligned} \tag{2.21}$$

For the total residual, we particularly obtain

$$\begin{aligned} r_N((v,q);\mu) &= r_N^1(v;\mu) + r_N^2(q;\mu) = a(e_N^u(\mu),v;\mu) + b(v,e_N^p(\mu);\mu) + b(e_N^u(\mu),q;\mu) \\ &\le \gamma_a(\mu)\|e_N^u(\mu)\|_X\|v\|_X + \gamma_b(\mu)\|v\|_X\|e_N^p(\mu)\|_Y + \gamma_b(\mu)\|e_N^u(\mu)\|_X\|q\|_Y \\ &= \begin{pmatrix}\|v\|_X \\ \|q\|_Y\end{pmatrix}^t \begin{pmatrix}\gamma_a(\mu) & \gamma_b(\mu) \\ \gamma_b(\mu) & 0\end{pmatrix} \begin{pmatrix}\|e_N^u(\mu)\|_X \\ \|e_N^p(\mu)\|_Y\end{pmatrix}, \quad \forall\,(v,q)\in Z; \end{aligned}$$

this may be further bounded in terms of the Euclidean norm $\|\cdot\|_2$,

$$\begin{aligned} &\le \|(v,q)\|_Z \left\| \begin{pmatrix}\gamma_a(\mu) & \gamma_b(\mu) \\ \gamma_b(\mu) & 0\end{pmatrix} \right\|_2 \|(e_N^u(\mu),e_N^p(\mu))\|_Z \\ &\le \big(\gamma_a(\mu)+\gamma_b(\mu)\big)\|e_N(\mu)\|_Z\|(v,q)\|_Z, \quad \forall\,(v,q)\in Z, \end{aligned}$$

where the last inequality follows from the relation $\|\cdot\|_2 \le \sqrt{\|\cdot\|_1\|\cdot\|_\infty}$ between matrix norms. Thus, the residual dual norms provide with

$$\beta_{\mathrm{Ba}}(\mu)\|e_N(\mu)\|_Z \le \|r_N(\cdot;\mu)\|_{Z'} \le \big(\gamma_a(\mu)+\gamma_b(\mu)\big)\|e_N(\mu)\|_Z$$

an *a posteriori* error estimator for $\|e_N(\mu)\|_Z$ that is not only reliable but also efficient (in the sense that the terms are used in the finite element community, i.e., bounding the error up to some constants from above and from below). The effectivities $\eta_N^{\mathrm{Ba}}(\mu) \equiv \Delta_N^{\mathrm{Ba}}(\mu)/\|e_N(\mu)\|_Z$ and $\eta_N^{\mathrm{Br}}(\mu) \equiv \Delta_N^{\mathrm{Br}}(\mu)/\|e_N(\mu)\|_Z$ satisfy

$$1 \le \eta_N^{\mathrm{Ba}}(\mu) \le \frac{\gamma_a(\mu)+\gamma_b(\mu)}{\beta_{\mathrm{Ba}}^{\mathrm{LB}}(\mu)}, \quad 1 \le \eta_N^{\mathrm{Br}}(\mu) \le C(\mu)\big(\gamma_a(\mu)+\gamma_b(\mu)\big),$$

where $C(\mu) > 0$ is a constant depending on $\gamma_a^{\mathrm{UB}}(\mu)$, $\alpha_a^{\mathrm{LB}}(\mu)$, and $\beta_{\mathrm{Br}}^{\mathrm{LB}}(\mu)$.

Second, it follows from (2.16), (2.18), and (1.8) that the error $\|e_N^u(\mu)\|_X$ is in fact bounded by

$$\|e_N^u(\mu)\|_X \le \frac{1}{\alpha_a(\mu)} \sup_{v\in\ker(B(\mu))} \frac{r_N^1(v;\mu)}{\|v\|_X} + \left(1+\frac{\gamma_a(\mu)}{\alpha_a(\mu)}\right)\frac{\|r_N^2(\cdot;\mu)\|_{Y'}}{\beta_{\mathrm{Br}}(\mu)}; \tag{2.22}$$

additionally using (2.21) and

$$\sup_{v\in\ker(B(\mu))} \frac{r_N^1(v;\mu)}{\|v\|_X} = \sup_{v\in\ker(B(\mu))} \frac{a(e_N^u(\mu),v;\mu)}{\|v\|_X} \le \gamma_a(\mu)\,\|e_N^u(\mu)\|_X,$$

we thus obtain a reliable and efficient *a posteriori* error estimator for $\|e_N^u(\mu)\|_X$ only.

2.3. Construction of Reduced Basis Approximation Spaces. We now turn to the construction of the RB approximation spaces X_N and Y_N. In general, the RB method constructs its low-dimensional approximation spaces by exploiting the parametric dependence of the problem: Solutions to the truth model problem (1.13) reside on the subset $\mathcal{M} \equiv \{\,(u(\mu),p(\mu)) \mid \mu\in\mathcal{D}\,\} \subset X\times Y$ and thus the method typically constructs $X_N\times Y_N$, $N\in\mathbb{N}_{\max}$, by focusing on $\mathcal{M}$. However, for saddle point problems, building the RB approximation space solely from snapshots $(u(\mu_N),p(\mu_N))\in\mathcal{M}$, $N\in\mathbb{N}_{\max}$, is not necessarily sufficient.

To explain this in greater detail, we first briefly recall some fundamental properties of the discrete system (2.1). Written in operator notation, it reads

$$A_N(\mu)\,u_N(\mu) + B_N(\mu)^t\,p_N(\mu) = f_N(\mu) \quad \text{in } X_N', \tag{2.23}$$

$$B_N(\mu)\,u_N(\mu) = g_N(\mu) \quad \text{in } Y_N', \tag{2.24}$$

where $A_N(\mu) : X_N \to X'_N$, $B_N(\mu) : X_N \to Y'_N$ and its transpose $B_N(\mu)^t$ are given by

$$\begin{aligned} \langle A_N(\mu)\, u_N, v_N\rangle &= a(u_N, v_N; \mu), && \forall\, u_N, v_N \in X_N, \\ \langle B_N(\mu)\, v_N, q_N\rangle &= b(v_N, q_N; \mu) = \langle B_N(\mu)^t\, q_N, v_N\rangle, && \forall\, v_N \in X_N,\ q_N \in Y_N; \end{aligned}$$

moreover, we denote $f_N(\mu) \equiv f(\cdot;\mu)|_{X_N} \in X'_N$ and $g_N(\mu) \equiv g(\cdot;\mu)|_{Y_N} \in Y'_N$.

For a given parameter $\mu \in \mathcal{D}$, the system (2.23), (2.24) is *solvable* if and only if $g_N(\mu)$ belongs to the range of the operator $B_N(\mu)$. In this case, we can construct $u_N(\mu) \in X_N$ satisfying (2.24) such that $f_N(\mu) - A_N(\mu)\, u_N(\mu)$ belongs to $\mathrm{im}(B_N(\mu)^t)$ and there exists $p_N(\mu) \in Y_N$ satisfying (2.23). The solution $u_N(\mu)$ is unique, and $p_N(\mu)$ is in general only determined up to an element of $\ker(B_N(\mu)^t)$. The system (2.23), (2.24) is therefore *uniquely solvable* if and only if $g_N(\mu)$ belongs to the range of the operator $B_N(\mu)$ *and* the operator $B_N(\mu)^t$ is injective. In the specific case of *finite*-dimensional spaces X_N and Y_N, $B_N(\mu)^t$ is injective if and only if $B_N(\mu)$ is surjective. Since this clearly implies $g_N \in \mathrm{im}(B_N(\mu))$ for *any* $g_N \in Y'_N$, we obtain: For a given parameter $\mu \in \mathcal{D}$, the system (2.23), (2.24) is well-posed if and only if $B_N(\mu)$ is surjective. This may be equivalently expressed by the inf-sup condition

$$\beta_N(\mu) \equiv \inf_{q_N \in Y_N} \sup_{v_N \in X_N} \frac{b(v_N, q_N; \mu)}{\|q_N\|_Y \|v_N\|_X} = \inf_{q_N \in Y_N} \frac{\|B_N(\mu)^t q_N\|_{X'_N}}{\|q_N\|_Y} > 0. \tag{2.25}$$

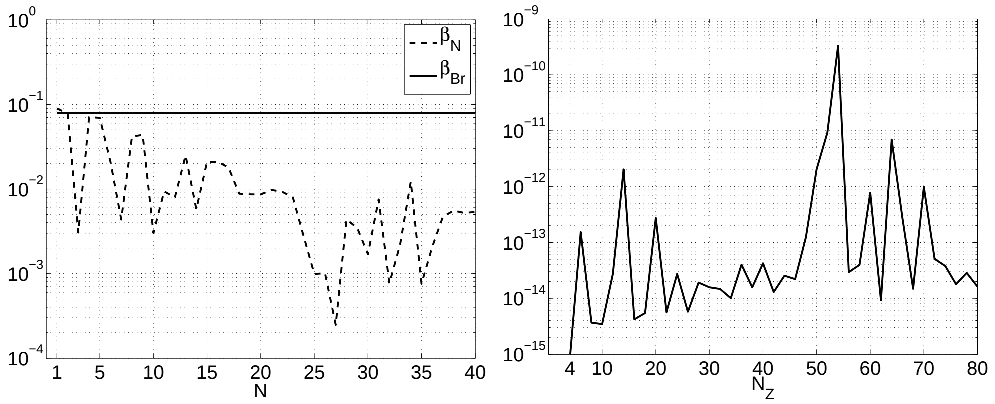
(For further details on any of the steps above, we refer to [4, 5].) For saddle point problems, the above inf-sup condition (2.25) thus represents an additional requirement for X_N, Y_N to provide useful approximations $u_N(\mu)$ and $p_N(\mu)$. In the remainder of this paper, a pair of approximation spaces (X_N, Y_N) satisfying (2.25) for *any* parameter value $\mu \in \mathcal{D}$ is called *stable*.

To demonstrate in practice that an RB approximation space constructed solely from snapshots $(u(\mu_N), p(\mu_N)) \in \mathcal{M}$, $N \in \mathbb{N}_{\max}$, may not provide useful approximations, we consider the following option: We assume that we are given a sample of parameter snapshots $\mathcal{D}_N \equiv \{\, \mu_n \mid 1 \le n \le N \,\} \subset \mathcal{D}$, $N \in \mathbb{N}_{\max}$; for our present purposes, $\mathcal{D}_N$ may represent any sequence of nested samples in $\mathcal{D}$, i.e., $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_N$. For $N \in \mathbb{N}_{\max}$, we then define our RB approximation spaces X_N and Y_N as

$$Y_N \equiv \mathrm{span}\{\, p(\mu_n) \mid 1 \le n \le N \,\} = \mathrm{span}\{\, \xi_n \mid 1 \le n \le N \,\}, \tag{2.26}$$

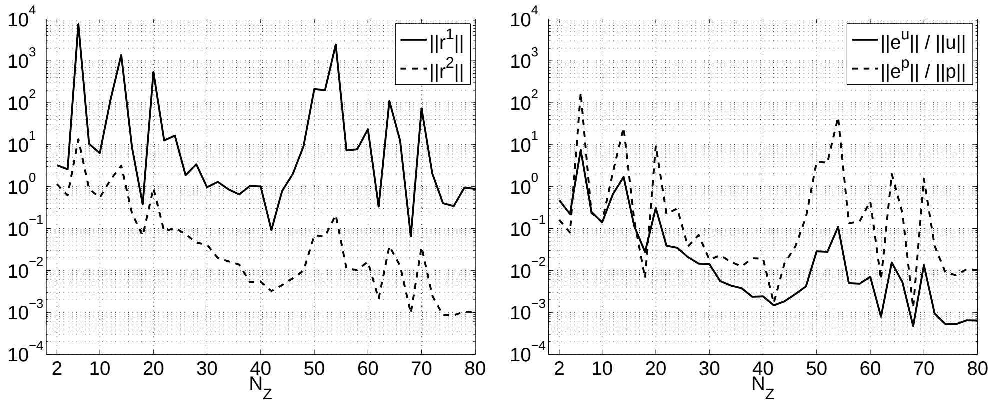
$$X_N^0 \equiv \mathrm{span}\{\, u(\mu_n) \mid 1 \le n \le N \,\}, \tag{2.27}$$

where $\xi_n \in Y$, $1 \le n \le N$, denote $(\cdot,\cdot)_Y$-orthonormal basis functions; we shall refer to this choice as Option 0. Let μ be arbitrary but fixed in $\mathcal{D}$. *A priori*, it is not known whether (X_N^0, Y_N) satisfies $\beta_N(\mu) > 0$: The system (2.23), (2.24) may very well be ill-posed. In practice, we obtain for our model problem (see §1.3) values for the inf-sup constants $\beta_N(\mu)$ that are small but still positive (see Fig. 2.1 (a)); the system (2.23), (2.24) is thus uniquely solvable and we obtain approximations $u_N(\mu)$ and $p_N(\mu)$ for $u(\mu)$ and $p(\mu)$, respectively. However, choosing the approximation spaces X_N, Y_N as in (2.26), (2.27) is in fact a special case: It is $\dim(X_N^0) = \dim(Y_N)$ and we thus observe a *locking phenomenon.* Positive inf-sup constants $\beta_N(\mu)$ provide a well-posed system but the operator $B_N(\mu)$ is not only surjective but bijective and $\ker(B_N(\mu))$ is trivial; therefore, there exists only one solution, $B_N(\mu)^{-1} g_N(\mu)$, satisfying the constraints (2.24) (see Fig. 2.1 (b)). Consequently, as we increase N, values of the residual dual norm $\|r_N^2(\cdot;\mu)\|_{Y'}$ associated with the constraints become relatively small and negligible in comparison to large values of $\|r_N^1(\cdot;\mu)\|_{X'}$ (see Fig. 2.2 (a)); our approximations $u_N(\mu)$ do not converge and thus, by (2.19), neither does $p_N(\mu)$ (see Fig. 2.2 (b)).



(a) (b)

FIG. 2.1. *Numerical results for $\mu = (0.1, 0.5)$ using Option 0 (2.27) based on a random sample $\mathcal{D}_N$ of parameter snapshots: (a) the inf-sup constants $\beta_N(\mu)$ (2.25) as a function of N and the truth inf-sup constant $\beta_{\mathrm{Br}}(\mu)$ (1.11); (b) the (absolute) distance $\|u_N(\mu) - B_N(\mu)^{-1} g_N(\mu)\|_X$ as a function of the total dimension N_Z.*



(a) (b)

FIG. 2.2. *Numerical results for $\mu = (0.1, 0.5)$ using Option 0 (2.27) based on a random sample $\mathcal{D}_N$ of parameter snapshots: (a) the residual dual norms $\|r_N^1(\cdot;\mu)\|_{X'}$, $\|r_N^2(\cdot;\mu)\|_{Y'}$ (see (2.5)) and (b) the relative errors $\|e_N^u(\mu)\|_X/\|u(\mu)\|_X$, $\|e_N^p(\mu)\|_Y/\|p(\mu)\|_Y$ (see (2.2)) as functions of the total dimension N_Z.*

It is shown in [29, 32] that the space X_N^0 can be enriched such that we obtain a *provably* stable pair (X_N, Y_N). For $\mu \in \mathcal{D}$ and $1 \le k \le Q_b$, let $T_\mu : Y \to X$ and $T^k : Y \to X$ be the *supremizer* functions (Riesz representers) given by

$$(T_\mu q, v)_X = b(v, q; \mu), \quad (T^k q, v)_X = b^k(v, q), \quad \forall\, v \in X,\ q \in Y; \tag{2.28}$$

here, $b^k(\cdot,\cdot)$, $1 \le k \le Q_b$, are the parameter-independent bilinear forms in the affine expansion of $b(\cdot,\cdot;\mu)$ (see (1.7)). For Y_N defined as in (2.26), we may then choose the space X_N as

$$X_N^1 \equiv X_N^0 \oplus \text{span}\{\, T^k \xi_n \mid 1 \le n \le N,\ 1 \le k \le Q_b \,\}; \tag{2.29}$$

we shall refer to this choice as Option 1. The pairs (X_N^1, Y_N), $N \in \mathbb{N}_{\max}$, then satisfy (see [29, 32])

$$\beta_N(\mu) \ge \beta_{\text{Br}}(\mu), \quad \forall\, \mu \in \mathcal{D}, \tag{2.30}$$

and (2.25) thus follows from the LBB inf-sup condition (1.11). Table 2.1 (a) contains values of the inf-sup constants $\beta_N(\mu)$ for several parameter values $\mu \in \mathcal{D}$: We clearly see that they are essentially constant with respect to N and indeed satisfy (2.30). However, Option 1 is also rather expensive since it requires $Q_b N$ additional basis functions compared to X_N^0. For this reason, we consider Option 2 (see [27, 32]): We here add only N supremizer functions instead of $Q_b N$ as in (2.29) to X_N^0,

$$X_N^2 \equiv X_N^0 \oplus \text{span}\{\, T_{\mu_n} \xi_n \mid 1 \le n \le N \,\}. \tag{2.31}$$

In this case, we can no longer prove stability of (X_N^2, Y_N) *a priori* but obtain significantly lower space dimensions than in Option 1. Table 2.1 (b) contains values of the inf-sup constants $\beta_N(\mu)$ for several parameter values $\mu \in \mathcal{D}$: We observe that they differ only insignificantly from the values obtained by using Option 1; they are also essentially constant in N and satisfy (2.30). In practice, Option 2 thus provides stable approximation spaces satisfying (2.30) at a fraction of the computational cost of Option 1 (see Fig. 2.3).

(a) Option 1

μ	$\beta_{\text{Br}}(\mu)$	$\beta_5(\mu)$	$\beta_{10}(\mu)$	$\beta_{20}(\mu)$	$\beta_{30}(\mu)$	$\beta_{40}(\mu)$
$(0.1, 0.1)$	$1.073 \cdot 10^{-1}$	$1.138 \cdot 10^{-1}$	$1.137 \cdot 10^{-1}$	$1.135 \cdot 10^{-1}$	$1.135 \cdot 10^{-1}$	$1.135 \cdot 10^{-1}$
$(0.5, 0.1)$	$1.070 \cdot 10^{-1}$	$1.134 \cdot 10^{-1}$	$1.133 \cdot 10^{-1}$	$1.132 \cdot 10^{-1}$	$1.132 \cdot 10^{-1}$	$1.131 \cdot 10^{-1}$
$(0.3, 0.3)$	$9.583 \cdot 10^{-2}$	$9.964 \cdot 10^{-2}$	$9.963 \cdot 10^{-2}$	$9.961 \cdot 10^{-2}$	$9.961 \cdot 10^{-2}$	$9.961 \cdot 10^{-2}$
$(0.1, 0.5)$	$7.881 \cdot 10^{-2}$	$8.097 \cdot 10^{-2}$	$8.088 \cdot 10^{-2}$	$8.085 \cdot 10^{-2}$	$8.084 \cdot 10^{-2}$	$8.083 \cdot 10^{-2}$
$(0.5, 0.5)$	$7.684 \cdot 10^{-2}$	$7.905 \cdot 10^{-2}$	$7.896 \cdot 10^{-2}$	$7.875 \cdot 10^{-2}$	$7.872 \cdot 10^{-2}$	$7.871 \cdot 10^{-2}$

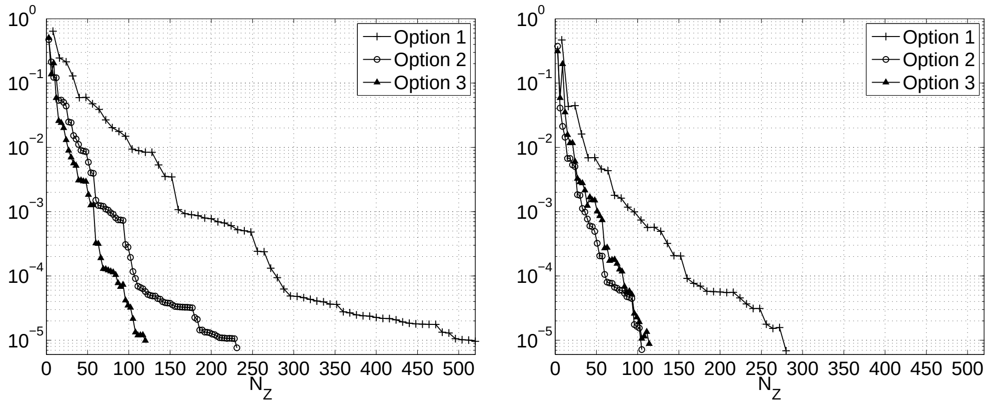
(b) Option 2

μ	$\beta_{\text{Br}}(\mu)$	$\beta_5(\mu)$	$\beta_{10}(\mu)$	$\beta_{20}(\mu)$	$\beta_{30}(\mu)$	$\beta_{40}(\mu)$
$(0.1, 0.1)$	$1.073 \cdot 10^{-1}$	$1.137 \cdot 10^{-1}$	$1.136 \cdot 10^{-1}$	$1.135 \cdot 10^{-1}$	$1.134 \cdot 10^{-1}$	$1.134 \cdot 10^{-1}$
$(0.5, 0.1)$	$1.070 \cdot 10^{-1}$	$1.134 \cdot 10^{-1}$	$1.133 \cdot 10^{-1}$	$1.132 \cdot 10^{-1}$	$1.131 \cdot 10^{-1}$	$1.131 \cdot 10^{-1}$
$(0.3, 0.3)$	$9.583 \cdot 10^{-2}$	$9.946 \cdot 10^{-2}$	$9.951 \cdot 10^{-2}$	$9.953 \cdot 10^{-2}$	$9.955 \cdot 10^{-2}$	$9.957 \cdot 10^{-2}$
$(0.1, 0.5)$	$7.881 \cdot 10^{-2}$	$8.034 \cdot 10^{-2}$	$8.040 \cdot 10^{-2}$	$8.049 \cdot 10^{-2}$	$8.059 \cdot 10^{-2}$	$8.072 \cdot 10^{-2}$
$(0.5, 0.5)$	$7.684 \cdot 10^{-2}$	$7.862 \cdot 10^{-2}$	$7.862 \cdot 10^{-2}$	$7.848 \cdot 10^{-2}$	$7.858 \cdot 10^{-2}$	$7.858 \cdot 10^{-2}$

TABLE 2.1

Values for the inf-sup constants $\beta_N(\mu)$ (2.25) for several parameter values $\mu \in \mathcal{D} = [0.1, 0.5]^2$ and N evaluated using (a) Option 1 (2.29) and (b) Option 2 (2.31) based on a random sample $\mathcal{D}_N$ of parameter snapshots.

The analysis and results for Option 0 clearly indicate the need to construct X_N, Y_N such that $\ker(B_N(\mu))$ is non-trivial. In Option 1 and Option 2, this has been achieved by an additional enrichment of X_N^0 with supremizer functions. For any parameter $\mu \in \mathcal{D}$, it can be expected from (2.22) in Remark 2.2 and also *a priori*



(a) (b)

Fig. 2.3. *Maximum relative errors (a)* $\|e^u_N(\mu)\|_X/\|u(\mu)\|_X$ *and (b)* $\|e^p_N(\mu)\|_Y/\|p(\mu)\|_Y$ *(see (2.2)) shown as functions of the total dimension* N_Z *computed using Option 1 (2.29), Option 2 (2.31), and Option 3 (2.34) based on random samples* $\mathcal{D}_N$ *and* $\mathcal{D}'_N$ *of parameter snapshots; the maximum is taken over 25 parameter values.*

error estimates (see [5]),

$$\begin{aligned} \|e^u_N(\mu)\|_X \le \left(1+\frac{\gamma_a(\mu)}{\alpha_a(\mu)}\right) \inf_{\substack{v_N\in X_N\\ B_N(\mu)v_N=g_N(\mu)}} \|u(\mu)-v_N\|_X \\ + \frac{\gamma_b(\mu)}{\alpha_a(\mu)} \inf_{q_N\in Y_N} \|p(\mu)-q_N\|_Y, \end{aligned} \tag{2.32}$$

$$\|e^p_N(\mu)\|_Y \le \left(1+\frac{\gamma_b(\mu)}{\beta_N(\mu)}\right) \inf_{q_N\in Y_N} \|p(\mu)-q_N\|_Y + \frac{\gamma_a(\mu)}{\beta_N(\mu)}\|e^u_N(\mu)\|_X, \tag{2.33}$$

that small inf-sup constants $\beta_N(\mu)$ will have more dramatic effects on $\|e^p_N(\mu)\|_Y$ than on $\|e^u_N(\mu)\|_X$: By (2.22) and (2.32), $\|e^u_N(\mu)\|_X$ clearly profits from a large space $\ker(B_N(\mu))$ but does not explicitly depend on the values of $\beta_N(\mu)$. Therefore, it is not surprising that adding supremizers as in Option 1 and Option 2, providing through (2.30) comparatively large values for the inf-sup constants $\beta_N(\mu)$, particularly favors the approximations for the Lagrange multiplier. Here, the RB approximations $p_N(\mu)$ for $p(\mu)$ converge much more rapidly than the approximations $u_N(\mu)$ for the primal variable $u(\mu)$ (see Fig. 2.3).

With the help of the above observations, we now aim to improve the RB approximations $u_N(\mu)$ for $u(\mu)$. The right-hand-side in (2.22) suggests that $\|e^u_N(\mu)\|_X$ first of all benefits from a good testing space $\ker(B(\mu))\cap X_N \subseteq \ker(B_N(\mu))$. For this purpose, we now enrich X^0_N with additional truth solutions $u(\mu')\in\ker(B(\mu'))$: Given a second sample of parameter snapshots $\mathcal{D}'_N \equiv \{\,\mu'_n \mid 1\le n\le N\,\}\subset\mathcal{D}$, $\mathcal{D}'_N\cap\mathcal{D}_N=\emptyset$, $N\in\mathbb{N}_{\max}$, we now define Option 3:

$$X^3_N \equiv X^0_N \oplus \operatorname{span}\{\,u(\mu'_n)\mid 1\le n\le N\,\}. \tag{2.34}$$

Again, we cannot prove *a priori* that this choice provides stable pairs (X^3_N, Y_N), $N\in\mathbb{N}_{\max}$. Figure 2.4 shows the relative distance $d^\beta_N(\mu)$ between $\beta_N(\mu)$ and the truth constant $\beta_{\mathrm{Br}}(\mu)$,

$$d^\beta_N(\mu) \equiv \max\left\{\frac{\beta_{\mathrm{Br}}(\mu)-\beta_N(\mu)}{\beta_{\mathrm{Br}}(\mu)}, 0\right\}, \quad \forall\,\mu\in\mathcal{D}. \tag{2.35}$$

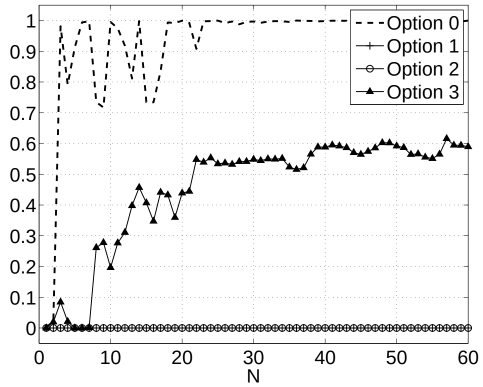


FIG. 2.4. *Maximum relative distance $d_N^\beta(\mu)$ (2.35) shown as a function of N computed using Option 0 (2.27), Option 1 (2.29), Option 2 (2.31), and Option 3 (2.34), respectively, based on random samples $\mathcal{D}_N$ and $\mathcal{D}'_N$ of parameter snapshots; the maximum is taken over 25 parameter values.*

A relative distance $d_N^\beta(\mu) < 1$ implies that $\beta_N(\mu)$ is positive; $d_N^\beta(\mu) > 0$ indicates that $\beta_N(\mu) < \beta_{\mathrm{Br}}(\mu)$. In case of Option 1 and Option 2, we thus obtain $d_N^\beta(\mu) = 0$ for all $\mu \in \mathcal{D}$, $N \in \mathbb{N}_{\max}$, due to (2.30). Option 3 provides in practice inf-sup constants $\beta_N(\mu)$ that are clearly positive but generally do not satisfy (2.30). However, compared to Option 1 and Option 2, approximations for the primal variable indeed improve considerably (see Fig. 2.3 (a)). Moreover, in (2.19) and (2.33), the smaller errors $\|e_N^u(\mu)\|_X$ seem to essentially compensate for the effects of the smaller inf-sup constants $\beta_N(\mu)$: Approximations for the Lagrange multiplier are not significantly worse (see Fig. 2.3 (b)).

As we shall often refer to them in §2.5, all introduced options are summarized at one glance in Table 2.2 together with their main respective properties.

Option	X_N	Special Properties
0	$X_N^0 \equiv \mathrm{span}\{\, u(\mu_n) \mid 1 \le n \le N \,\}$	$\dim(X_N) = \dim(Y_N)$ $\beta_N(\mu) > 0$ with $d_N^\beta(\mu)$ large
1	$X_N^1 \equiv X_N^0 \oplus \mathrm{span}\{\, T^k \xi_n \mid 1 \le n \le N,\ 1 \le k \le Q_b \,\}$	$\beta_N(\mu) \ge \beta_{\mathrm{Br}}(\mu)$ (in theory and practice)
2	$X_N^2 \equiv X_N^0 \oplus \mathrm{span}\{\, T_{\mu_n} \xi_n \mid 1 \le n \le N \,\}$	$\beta_N(\mu) \ge \beta_{\mathrm{Br}}(\mu)$ (in practice) $\dim(X_N^2) \ll \dim(X_N^1)$
3	$X_N^3 \equiv X_N^0 \oplus \mathrm{span}\{\, u(\mu'_n) \mid 1 \le n \le N \,\}$	good space $\ker(B_N(\mu))$ $\beta_N(\mu) > 0$ with $d_N^\beta(\mu)$ small

TABLE 2.2
Options for the construction of the RB approximation space X_N for the primal variable, where the RB approximation space Y_N for the Lagrange multiplier is given as in (2.26).

REMARK 2.4. Note that for algebraic stability reasons, we in fact express both Y_N and X_N by orthonormal basis functions: According to (2.26), we also set $X_N^i = \mathrm{span}\{\, \phi_n^i \mid 1 \le n \le N_X^i \,\}$, where $\phi_n^i \in X$, $1 \le n \le N_X^i$, are $(\cdot,\cdot)_X$-orthonormal, $i = 0, 1, 2, 3$.

2.4. Offline–Online Computational Procedure. The basic strategy lies in the μ-affine dependence of the involved operators. All μ-independent quantities in (1.7) can be formed and stored within a computationally expensive Offline stage, which is performed only once and the cost of which depends on the large finite element dimension $\mathcal{N}$. For any given parameter $\mu \in \mathcal{D}$, the RB approximation is then computed within a highly efficient Online stage; the cost does not depend on $\mathcal{N}$ but only on the considerably smaller dimension of the RB approximation space. Since much of this machinery is by now standard in RB methods, we refer to [26, 31] for technical details on the assembly of the system (2.1).

It is clear that there are two components to the computation of the *a posteriori* error bounds $\Delta_N^{\mathrm{Ba}}(\mu)$, $\Delta_N^u(\mu)$, $\Delta_N^p(\mu)$, and $\Delta_N^{\mathrm{Br}}(\mu)$: The calculation of the residual dual norms (2.5), and the calculation of the required lower and upper bounds to the coercivity, continuity, or inf-sup stability constants (see (2.3), (2.4)). The former is again an application of now standard RB techniques that can be found in, e.g., [10, 31]. The latter is achieved by a successive constraint method (SCM) proposed by Huynh, et al. in [15]. For exact configurations and computational details to our model probem, we refer to §2.6.

REMARK 2.5. Note that there exist several approaches for the computation of lower bounds to coercivity and inf-sup stability constants. The applicability of techniques proposed in [21, 26] is unfortunately very restricted (e.g., to the case of positive coefficient functions in (1.7)); for problems involving geometry variations, we rely on more general but also more complicated and Offline-expensive approaches. Compared to earlier proposals [25, 33, 37], the SCM in [15] represents a very generally applicable method that performs better and is easier to implement. In addition to accurate lower bounds, it also provides upper bounds that are remarkably sharp and are thus valuable estimates for the respective coercivity and inf-sup constants. Nevertheless, for inf-sup stability constants, it still involves notable Offline computations that depend on a Q^2-term affine parameter expansion (see also §2.6). This has been tackled by more recent techniques based on a "natural norm" [14]. However, though well-suited for certain problems, the algorithm is in our situation highly cumbersome and involves eigenvalue problems that are much more difficult to solve; also, no upper bounds are provided.

2.5. Adaptive Sampling Procedures. In this section, all parts of the methodology discussed in §2.2, §2.3, and §2.4 coalesce to tackle the key question of how to construct RB approximation spaces providing rapidly convergent approximations.

Rigorous and computationally efficient *a posteriori* RB error bounds enable us to invoke a greedy sampling process (see [3, 6] and references therein) identifying relatively few basis functions that suffice to warrant a desired accuracy for any parameter query: Starting with an exhaustive sample Σ of parameter points in $\mathcal{D}$ and an initial sample μ_1, it detects the parameter μ_2 for which the error bound attains its maximum over Σ, and then enriches the current RB approximation spaces with the new basis functions associated with μ_2 (see §2.3); this step is repeated until a prescribed error tolerance is satisfied. In the case of saddle point problems, the inf-sup condition (2.25) represents an additional requirement for the RB approximation spaces. Therefore, according to our observations in §2.3, the above sampling process will in general (e.g., in combination with Option 0 (2.27)) not succeed. In earlier work [8, 10, 23], it has been applied in combination with either Option 1 (2.29) or Option 2 (2.31), where stability is guaranteed by adding supremizer functions to the RB approximation space X_N for the primal variable *in each step*. For Option 2, the exact procedure is summarized in Algorithm 1.

However, using this procedure, the question arises whether the resulting RB approximation spaces are indeed computationally efficient and an RB approximation

Algorithm 1 Standard Greedy Algorithm (using Option 2 (2.31) and $\Delta_N^u(\mu)$ (2.11))

Choose $\Sigma \subset \mathcal{D}$, $\delta_{\text{tol}} \in (0,1)$, and $\mu_1 \in \Sigma$
Set $N \leftarrow 0$, $\mathcal{D}_N \leftarrow \{\}$, $N_Y \leftarrow 0$, $Y_N \leftarrow \{\}$, $N_X \leftarrow 0$, $X_N \leftarrow \{\}$
repeat
    $N \leftarrow N+1$, $\mathcal{D}_N \leftarrow \mathcal{D}_{N-1} \cup \{\mu_N\}$
    $N_Y \leftarrow N_Y + 1$, $Y_N \leftarrow Y_{N-1} \oplus \text{span}\{\, p(\mu_N)\,\} = Y_{N-1} \oplus \text{span}\{\, \xi_N \,\}$ (see (2.26))
    $N_X \leftarrow N_X + 2$, $X_N \leftarrow X_{N-1} \oplus \text{span}\{\, u(\mu_N), T_{\mu_N}\xi_N \,\}$ (see (2.31))
    for all $\mu \in \Sigma$ **do**
        Compute $(u_N(\mu), p_N(\mu))$ and $\Delta_N(\mu) \equiv \frac{\Delta_N^u(\mu)}{\|u_N(\mu)\|_X}$
    end for
    $\mu_{N+1} \equiv \arg\max_{\mu\in\Sigma} \; \Delta_N(\mu)$
until $\Delta_N(\mu_{N+1}) < \delta_{\text{tol}}$
$N_{\max} \leftarrow N$

Algorithm 2

Choose $\Sigma \subset \mathcal{D}$, $\delta_{\text{tol}}, \delta_{\text{tol}}^{\beta} \in (0,1)$, and $\mu_1 \in \Sigma$
Set $N \leftarrow 0$, $\mathcal{D}_N \leftarrow \{\}$, $N_Y \leftarrow 0$, $Y_N \leftarrow \{\}$, $N_X \leftarrow 0$, $X_N \leftarrow \{\}$
repeat
    $N \leftarrow N+1$, $\mathcal{D}_N \leftarrow \mathcal{D}_{N-1} \cup \{\mu_N\}$
    $N_Y \leftarrow N_Y + 1$, $Y_N \leftarrow Y_{N-1} \oplus \text{span}\{\, p(\mu_N)\,\}$
    $N_X \leftarrow N_X + 1$, $X_N \leftarrow X_{N-1} \oplus \text{span}\{\, u(\mu_N)\,\}$
    while (true) do
        for all $\mu \in \Sigma$ **do**
            Compute $\hat{d}_N^{\beta}(\mu) \equiv \max\left\{ \frac{\beta_{\text{Br}}^{\text{UB}}(\mu) - \beta_N(\mu)}{\beta_{\text{Br}}^{\text{UB}}(\mu)}, 0 \right\}$
        end for
        $\mu^* \equiv \arg\max_{\mu\in\Sigma} \hat{d}_N^{\beta}(\mu)$
        if $\hat{d}_N^{\beta}(\mu^*) < \delta_{\text{tol}}^{\beta}$, **then**
            break
        end if
        $N_X \leftarrow N_X + 1$, $X_N \leftarrow X_N \oplus \text{span}\{\, T_{\mu^*}\varrho_N(\mu^*)\,\}$
    end while
    for all $\mu \in \Sigma$ **do**
        Compute $(u_N(\mu), p_N(\mu))$ and $\Delta_N(\mu) \equiv \frac{\Delta_N^u(\mu)}{\|u_N(\mu)\|_X}$
    end for
    $\mu_{N+1} \equiv \arg\max_{\mu\in\Sigma} \; \Delta_N(\mu)$
until $\Delta_N(\mu_{N+1}) < \delta_{\text{tol}}$
$N_{\max} \leftarrow N$

space X_N with a dimension twice as large as for Y_N is required to guarantee stability. Addressing this, we now present a new adaptive sampling procedure for saddle point problems: In addition to greedily selecting the parameter snapshots $\mu_1, \ldots, \mu_N$ (see §2.3), we now also adaptively recognize the need for stabilization and thus identify relatively few basis functions that suffice to warrant *both* a desired accuracy *and* stability for any parameter query.

We assume that we are given a stable pair (X_{N-1}, Y_{N-1}) of RB approximation spaces that is based on the parameter snapshots $\mathcal{D}_{N-1} \equiv \{\, \mu_1, \ldots, \mu_{N-1}\,\} \subset \Sigma$ (see §2.3). As in the standard greedy procedure (see Algorithm 1), we use our rigorous

Algorithm 3

Choose $\Sigma \subset \mathcal{D}$, $\delta_{\rm tol}, \delta_{\rm tol}^{\beta} \in (0,1)$, and $\mu_1 \in \Sigma$
Set $N \leftarrow 0$, $\mathcal{D}_N \leftarrow \{\}$, $\mathcal{D}' \leftarrow \{\}$, $N_Y \leftarrow 0$, $Y_N \leftarrow \{\}$, $N_X \leftarrow 0$, $X_N \leftarrow \{\}$
repeat
  $N \leftarrow N+1$, $\mathcal{D}_N \leftarrow \mathcal{D}_{N-1} \cup \{\mu_N\}$
  $N_Y \leftarrow N_Y + 1$, $Y_N \leftarrow Y_{N-1} \oplus \mathrm{span}\{\, p(\mu_N)\,\}$
  if $\mu_N \notin \mathcal{D}'$, **then**
    $N_X \leftarrow N_X + 1$, $X_N \leftarrow X_{N-1} \oplus \mathrm{span}\{\, u(\mu_N)\,\}$
  end if
  while (**true**) **do**
    for all $\mu \in \Sigma$ **do**
      Compute $(u_N(\mu), p_N(\mu))$, $\Delta_N(\mu) \equiv \frac{\Delta_N^u(\mu)}{\|u_N(\mu)\|_X}$
      Compute $\hat{d}_N^{\beta}(\mu) \equiv \max\left\{ \frac{\beta_{\rm Br}^{\rm UB}(\mu) - \beta_N(\mu)}{\beta_{\rm Br}^{\rm UB}(\mu)}, 0 \right\}$
    end for
    $\mu_N' \equiv \arg\max_{\mu\in\Sigma}\ \Delta_N(\mu)$, $\mu^* \equiv \arg\max_{\mu\in\Sigma} \hat{d}_N^{\beta}(\mu)$
    if $\hat{d}_N^{\beta}(\mu^*) < \delta_{\rm tol}^{\beta}$, **then**
      $\mu_{N+1} \equiv \mu_N'$
      break
    end if
    $N_X \leftarrow N_X + 1$
    if $\min_{\mu\in\mathcal{D}'\cup\mathcal{D}_N} \frac{|\mu_N' - \mu|}{|\mu|} \geq 0.1\%$, **then**
      $\mathcal{D}' \leftarrow \mathcal{D}' \cup \{\mu_N'\}$
      $X_N \leftarrow X_N \oplus \mathrm{span}\{\, u(\mu_N')\,\}$
    else
      $X_N \leftarrow X_N \oplus \mathrm{span}\{\, T_{\mu^*} \varrho_N(\mu^*)\,\}$
    end if
  end while
until $\Delta_N(\mu_{N+1}) < \delta_{\rm tol}$
$N_{\max} \leftarrow N$

and (Online-)efficient error bounds (see §2.2 and §2.4) to find $\mu_N \in \Sigma$ and thus the truth solution $(u(\mu_N), p(\mu_N)) \in \mathcal{M}$ that is represented least in the current basis; this is then appended to the current basis and we obtain a subsequent pair (X_N, Y_N).

From our observations in §2.3, we see that the new pair (X_N, Y_N) will in general only provide useful approximations if the inf-sup constants $\beta_N(\mu)$, $\mu \in \mathcal{D}$, do not become too small. This can be achieved through an additional enrichment of X_N: We may add (i) supremizer functions as in (2.31), favoring the Lagrange multiplier, or (ii) additional truth solutions as in (2.34), favoring the primal variable. We note from (2.25) that the inf-sup constants $\beta_N(\mu)$ may be computed as the solution to a low-dimensional eigenvalue problem; the associated eigenvector is denoted by $\varrho_N(\mu) \in Y_N$,

$$\varrho_N(\mu) \equiv \arg \inf_{q_N \in Y_N} \frac{\|B_N(\mu)^t q_N\|_{X_N'}}{\|q_N\|_Y}, \quad \forall\, \mu\ \in \mathcal{D}. \tag{2.36}$$

Whether the pair (X_N, Y_N) needs to be stabilized is now indicated by the distance $d_N^{\beta}(\mu)$ (2.35): We find the parameter $\mu^* \in \Sigma$ where $d_N^{\beta}(\mu)$ attains its maximum and whenever $d_N^{\beta}(\mu^*)$ exceeds a prescribed tolerance $\delta_{\rm tol}^{\beta} \in (0,1)$, we additionally enrich the space X_N.

First, in Algorithm 2, we enrich X_N with the supremizer function $T_{\mu^*}\varrho_N(\mu^*) \in X$ (see (2.28)), $X_N^+ \equiv X_N \oplus \text{span}\{\,T_{\mu^*}\varrho_N(\mu^*)\,\}$. With $X_N^+ \supset X_N$, the enriched pair (X_N^+, Y_N) then clearly satisfies

$$\beta_N^+(\mu) = \inf_{q_N \in Y_N} \sup_{v_N \in X_N^+} \frac{b(v_N, q_N; \mu)}{\|q_N\|_Y \|v_N\|_X} \geq \inf_{q_N \in Y_N} \sup_{v_N \in X_N} \frac{b(v_N, q_N; \mu)}{\|q_N\|_Y \|v_N\|_X} = \beta_N(\mu),$$

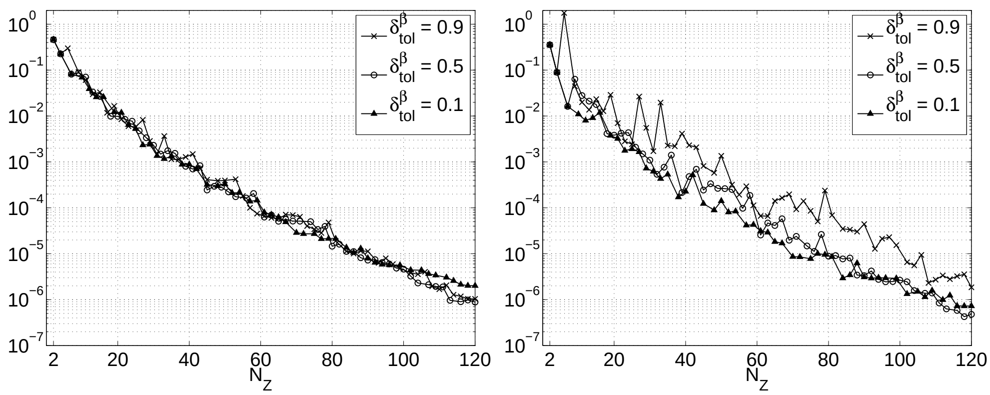
for all $\mu \in \mathcal{D}$. From (2.28), it moreover follows that

$$\frac{\|B_N^+(\mu^*)^t \varrho_N(\mu^*)\|_{(X_N^+)'}}{\|\varrho_N(\mu^*)\|_Y} = \frac{\|B(\mu^*)^t \varrho_N(\mu^*)\|_{X'}}{\|\varrho_N(\mu^*)\|_Y} \geq \beta_{\text{Br}}(\mu^*),$$

where the inequality holds by definition of $\beta_{\text{Br}}(\mu)$ in (1.11). As long as $d_N^{\beta,+}(\mu^*) > 0$, we therefore obtain $\varrho_N^+(\mu^*) \neq \varrho_N(\mu^*)$ and thus $\beta_N^+(\mu^*) > \beta_N(\mu^*)$ whenever $\beta_N(\mu^*)$ is a simple eigenvalue. Second, in Algorithm 3, we aim to stabilize the pair (X_N, Y_N) by enriching X_N with additional truth solutions $u(\mu_N')$. Only if this is not possible do we resort to the supremizer functions as in Algorithm 2.

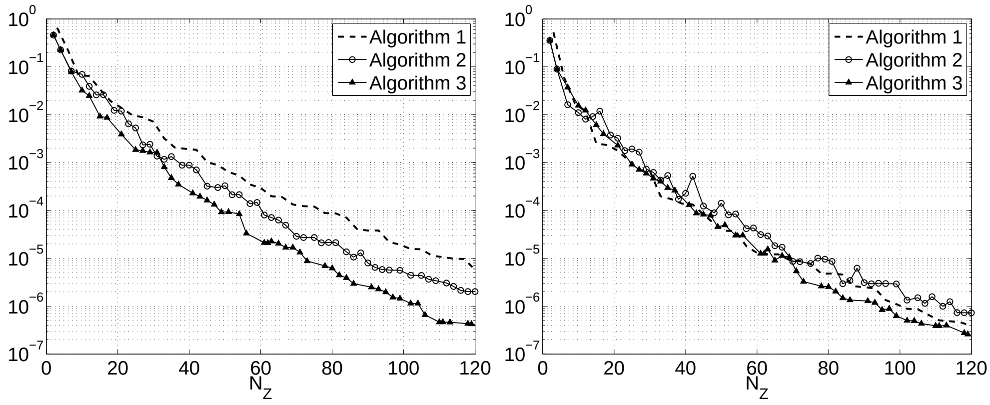
Figure 2.5 now presents numerical results for Algorithm 2 using different values for $\delta_{\text{tol}}^\beta \in (0,1)$; a direct comparison with Algorithm 3 is then given in Fig. 2.6. Again, it is verified in practice what *a posteriori* and *a priori* error bounds (2.22), (2.32), and (2.33) indicate: Small inf-sup constants $\beta_N(\mu)$ have much stronger effects on $\|e_N^p(\mu)\|_Y$ than on $\|e_N^u(\mu)\|_X$. Even for fairly large tolerances $\delta_{\text{tol}}^\beta$, we obtain approximation spaces X_N, Y_N providing very accurate approximations $u_N(\mu)$ that do not necessarily further improve for lower values of $\delta_{\text{tol}}^\beta$ (see Fig. 2.5 (a)). In contrast, accurate approximations for the Lagrange multiplier require a smaller tolerance $\delta_{\text{tol}}^\beta$ and improve considerably in dependence of decreasing values of $\delta_{\text{tol}}^\beta$ (see Fig. 2.5 (b)). The errors $\|e_N^u(\mu)\|_X$ primarily benefit from a good testing space $\ker(B(\mu)) \cap X_N \subseteq \ker(B_N(\mu))$ (see §2.3) as achieved in Algorithm 3 (see Fig. 2.6 (a)).

Figure 2.6 moreover demonstrates that the standard greedy procedure (see Algorithm 1) may indeed generally be inefficient: For our model problem (see §1.3), Algorithm 2 provides accurate approximations for the primal variable at much less



(a) (b)

FIG. 2.5. *Numerical results for Algorithm 2 with $\delta_{\text{tol}}^\beta \in \{0.9, 0.5, 0.1\}$ and a random sample Σ of size $|\Sigma| = 4{,}900$: Maximum relative errors (a) $\|e_N^u(\mu)\|_X / \|u(\mu)\|_X$ and (b) $\|e_N^p(\mu)\|_Y / \|p(\mu)\|_Y$ (see (2.2)) are shown as functions of N_Z; the maximum is taken over 25 parameter values.*



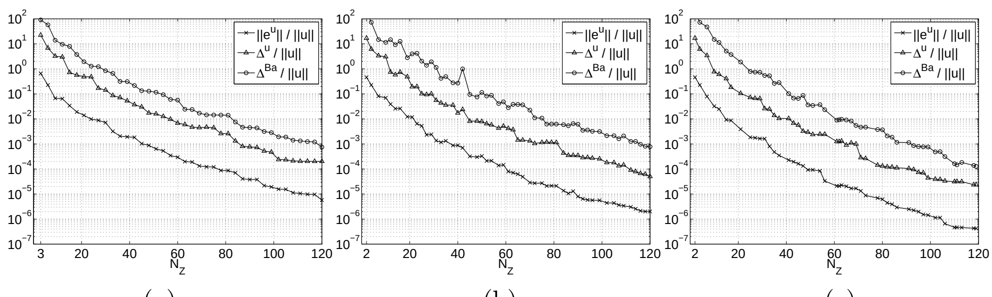
(a) (b)

FIG. 2.6. *Comparison of Algorithm 2 and Algorithm 3 (both using* $\delta_{\rm tol}^{\beta} = 0.1$*) to the standard greedy procedure in Algorithm 1, using a random sample* Σ *of size* $|\Sigma| = 4{,}900$*: Maximum relative errors (a)* $\|e_N^u(\mu)\|_X/\|u(\mu)\|_X$ *and (b)* $\|e_N^p(\mu)\|_Y/\|p(\mu)\|_Y$ *(see (2.2)) are shown as functions of* N_Z*; the maximum is taken over 25 parameter values.*

computational cost; using Algorithm 3, we obtain even greater computational savings compared to Algorithm 1 of up to 35%.

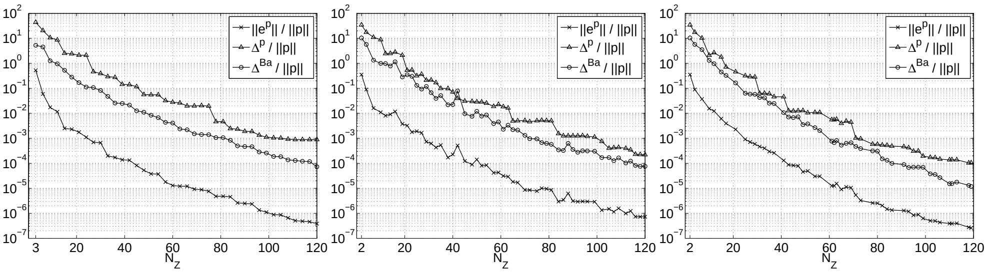
2.6. Performance of *A Posteriori* Error Bounds. In the previous section, we have seen how different constructions of the RB approximation spaces affect the convergence of the RB approximations. However, as the associated truth solutions are usually not computed, the errors in our RB approximations are unknown quantities that may only be estimated by *a posteriori* error bounds. We now comment on the sharpness and computational efficiency of the rigorous *a posteriori* error bounds presented in §2.2.

First, we compare $\Delta_N^u(\mu)$, $\Delta_N^p(\mu)$ and $\Delta_N^{\rm Ba}(\mu)$ with respect to sharpness. We construct the RB approximation spaces by the sampling processes Algorithm 1, Algorithm 2, and Algorithm 3 presented in §2.5 based on an exhaustive random sample Σ of size $|\Sigma| = 4{,}900$. Fig. 2.7 and Fig. 2.8 show the maximum errors in the RB



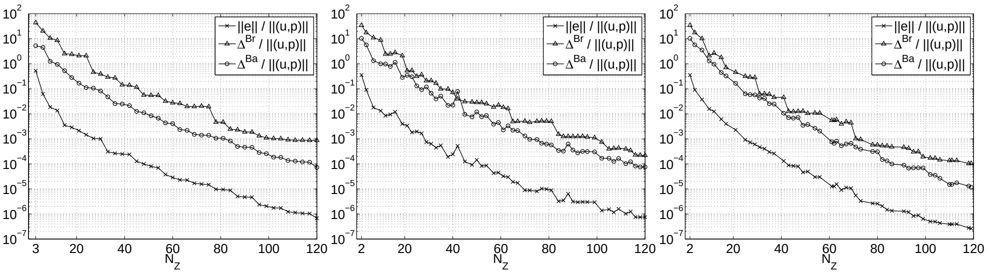
(a) (b) (c)

FIG. 2.7. *Maximum relative error* $\|e_N^u(\mu)\|_X/\|u(\mu)\|_X$ *and maximum relative error bounds* $\Delta_N^u(\mu)/\|u(\mu)\|_X$, $\Delta_N^{\rm Ba}(\mu)/\|u(\mu)\|_X$ *(see (2.2), (2.11), and (2.9)) shown as functions of* N_Z *for (a) Algorithm 1, (b) Algorithm 2 with* $\delta_{\rm tol}^{\beta} = 0.1$*, and (c) Algorithm 3 with* $\delta_{\rm tol}^{\beta} = 0.1$ *(see §2.5); the maximum is taken over 25 parameter values; the computation of the error bounds is based on the exact constants* $\alpha_a(\mu)$*,* $\gamma_a(\mu)$*,* $\beta_{\rm Br}(\mu)$*, and* $\beta_{\rm Ba}(\mu)$*.*



(a) (b) (c)

FIG. 2.8. *Maximum relative error* $\|e_N^p(\mu)\|_Y/\|p(\mu)\|_Y$ *and maximum relative error bounds* $\Delta_N^p(\mu)/\|p(\mu)\|_Y$, $\Delta_N^{\rm Ba}(\mu)/\|p(\mu)\|_Y$ *(see (2.2), (2.12), and (2.9)) shown as functions of* N_Z *for (a) Algorithm 1, (b) Algorithm 2 with* $\delta_{\rm tol}^{\beta} = 0.1$, *and (c) Algorithm 3 with* $\delta_{\rm tol}^{\beta} = 0.1$ *(see §2.5); the maximum is taken over 25 parameter values; the computation of the error bounds is based on the exact constants* $\alpha_a(\mu)$, $\gamma_a(\mu)$, $\beta_{\rm Br}(\mu)$, *and* $\beta_{\rm Ba}(\mu)$.



(a) (b) (c)

FIG. 2.9. *Maximum relative error* $\|e_N(\mu)\|_Z/\|(u(\mu),p(\mu))\|_Z$ *and maximum relative error bounds* $\Delta_N^{\rm Br}(\mu)/\|(u(\mu),p(\mu))\|_Z$, $\Delta_N^{\rm Ba}(\mu)/\|(u(\mu),p(\mu))\|_Z$ *(see (2.2), (2.20), and (2.9)) shown as functions of* N_Z *for (a) Algorithm 1, (b) Algorithm 2 with* $\delta_{\rm tol}^{\beta} = 0.1$, *and (c) Algorithm 3 with* $\delta_{\rm tol}^{\beta} = 0.1$ *(see §2.5); the maximum is taken over 25 parameter values; the computation of the error bounds is based on the exact constants* $\alpha_a(\mu)$, $\gamma_a(\mu)$, $\beta_{\rm Br}(\mu)$, *and* $\beta_{\rm Ba}(\mu)$.

approximations for the primal variable and the Lagrange multiplier, respectively, together with the respective error bounds (2.9) and (2.11), (2.12); Fig. 2.9 shows the maximum total error in the RB approximation and associated error bounds (2.9) and (2.20). The maximum is computed over a sample of 25 parameter values. For this sample, to analyze only the effects of the *a posteriori* error bound formulations and eliminate contributions of the SCM, the exact constants $\alpha_a(\mu)$, $\gamma_a(\mu)$, $\beta_{\rm Br}(\mu)$, and $\beta_{\rm Ba}(\mu)$ rather than the lower/upper bounds (2.3), (2.4) are used. Effectivities associated with the error bounds are given in Fig. 2.10. With maximum effectivities ranging from 50 to 90, $\Delta_N^u(\mu)$ represents a reasonably sharp bound for the error in the RB approximation $u_N(\mu)$. In contrast, $\Delta_N^p(\mu)$ estimates the error in $p_N(\mu)$ rather pessimistically. Exact effectivities here strongly depend on how the underlying RB approximation spaces are constructed: Using Algorithm 1 (see Fig. 2.10 (a)), we obtain effectivity values of order 10^3; in this case, $\Delta_N^{\rm Ba}(\mu)$ clearly represents a much sharper bound. However, using Algorithm 2 or Algorithm 3 (see Fig. 2.10 (b) and (c)), the difference is not as large.

We now turn to the computational efficiency of $\Delta_N^u(\mu)$, $\Delta_N^p(\mu)$ and $\Delta_N^{\rm Ba}(\mu)$, respectively. The SCM (see §2.4) enables the (Online-)efficient estimation of the

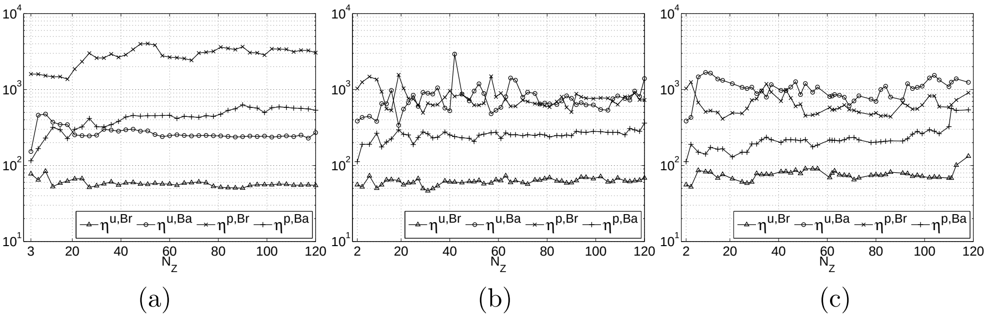


FIG. 2.10. *Maximum effectivities* $\eta_N^{u,\mathrm{Br}}(\mu) \equiv \Delta_N^u(\mu)/\|e_N^u(\mu)\|_X$*,* $\eta_N^{u,\mathrm{Ba}}(\mu) \equiv \Delta_N^{\mathrm{Ba}}(\mu)/\|e_N^u(\mu)\|_X$*,* $\eta_N^{p,\mathrm{Br}}(\mu) \equiv \Delta_N^p(\mu)/\|e_N^p(\mu)\|_Y$*, and* $\eta_N^{p,\mathrm{Ba}}(\mu) \equiv \Delta_N^{\mathrm{Ba}}(\mu)/\|e_N^p(\mu)\|_Y$ *(see (2.2), (2.11), (2.12), and (2.9)) for (a) Algorithm 1, (b) Algorithm 2 with* $\delta_{\mathrm{tol}}^{\beta} = 0.1$*, and (c) Algorithm 3 with* $\delta_{\mathrm{tol}}^{\beta} = 0.1$ *(see §2.5); effectivities are shown as functions of* N_Z*; the maximum is taken over 25 parameter values; the computation of the error bounds is based on the exact constants* $\alpha_a(\mu)$*,* $\gamma_a(\mu)$*,* $\beta_{\mathrm{Br}}(\mu)$*, and* $\beta_{\mathrm{Ba}}(\mu)$*. The effectivities* $\eta_N^{\mathrm{Br}}(\mu)$ *and* $\eta_N^{\mathrm{Ba}}(\mu)$ *(see Remark 2.2) are not listed separately as they behave similarly to* $\eta_N^{p,\mathrm{Br}}(\mu)$ *and* $\eta_N^{p,\mathrm{Ba}}(\mu)$*, respectively.*

constants $\gamma_a(\mu)$, $\alpha_a(\mu)$, and $\beta_{\mathrm{Br}}(\mu)$. Using the terminology of [15], we apply the method for $M_\alpha = \infty$, $M_+ = 0$, and an exhaustive sample Ξ of size $J = 4225$. We set the SCM tolerance $\varepsilon = 0.01$ (resp., 0.5) and obtain $K_{\max} = 12$ (resp., 5) for the continuity constants $\gamma_a(\mu)$ and $K_{\max} = 35$ (resp., 6) for the coercivity constants $\alpha_a(\mu)$. For the Brezzi inf-sup constants $\beta_{\mathrm{Br}}(\mu)$, we set $\varepsilon = 0.5$ (resp., 0.75) and obtain $K_{\max} = 163$ (resp., 111). Based on these computations, we receive very accurate (Online-)efficient bounds $\alpha_a^{\mathrm{LB}}(\mu)$, $\gamma_a^{\mathrm{UB}}(\mu)$, and $\beta_{\mathrm{Br}}^{\mathrm{LB}}(\mu)$ (see (2.3), (2.4)), providing *a posteriori* error bounds $\Delta_N^u(\mu)$, $\Delta_N^p(\mu)$ that essentially coincide with their values based on the evaluation of the exact constants; associated effectivities thus remain the same as shown in Fig. 2.10.

Using a 2.26 GHz Intel Core 2 Duo processor, Offline computations necessary for the SCM applied to the Babuška inf-sup constants $\beta_{\mathrm{Ba}}(\mu)$ are infeasible. The difficulty here lies in the computations required for the SCM bounding box $\mathcal{B}_Q$ (see [15]): The differential operator associated with the Babuška inf-sup constants exhibits a μ-affine decomposition (1.7) that involves Q terms where $\max\{Q_a, Q_b\} \le Q \le Q_a + Q_b$; the SCM bounding box then requires the solution of $Q(1+Q)$ generalized eigenvalue problems. (Recall that in our model problem, we have $Q_a = 10$ and $Q_b = 6$ (see §1.3); here, $Q = 15$.) However, even the solution of a *single* generalized eigenvalue problem associated with $\mathcal{B}_Q$ is a computational challenge: System matrices on the left-hand-side are dense, symmetric, not necessarily positive definite, and of a very low rank due to the geometric transformations involved (see §1.3). Consequently, even for our model problem where geometric variations are still relatively simple, we would have to resort to one of the other approaches discussed in §2.4 to obtain (Online-)efficient lower bounds to the inf-sup constants $\beta_{\mathrm{Ba}}(\mu)$ (see (2.4)). No matter which of the currently existing methods is used though, the (Online-)efficient evaluation of $\Delta_N^{\mathrm{Ba}}(\mu)$ requires either Offline costs that are prohibitively expensive compared to the ones needed for $\Delta_N^u(\mu)$, $\Delta_N^p(\mu)$, or a significant loss in terms of accuracy causing associated effectivities to be much worse than in Fig. 2.10.

We may now discuss the Online computation times for the new proposed methods. For comparison, once the μ-independent parts in (1.7) have been formed, direct evaluation of the truth approximation $(u(\mu), p(\mu))$ (i.e., assembly and solution of (1.13)) takes 7.5 seconds. The rigorous and efficient error bounds $\Delta_N^u(\mu)$ and $\Delta_N^p(\mu)$ allow us to choose the RB system dimension N_Z just large enough to obtain a desired accuracy. In case of Algorithm 1, we need $N_Z = 57$ to achieve a prescribed accuracy of roughly 1% or better in the RB approximations $u_N(\mu)$ (see Fig. 2.7 (a)). Once the database has been loaded, the Online calculation of $(u_N(\mu), p_N(\mu))$ (i.e., assembly and solution of (2.1)) and $\Delta_N^u(\mu)$, $\Delta_N^p(\mu)$ for any new value of μ takes 0.8 and 24.6 milliseconds, respectively, which is in total roughly 290× faster than direct evaluation of the truth approximation. In the case of Algorithm 3, the same accuracy is achieved for $N_Z = 43$ (see Fig. 2.7 (c)); the Online calculation of $(u_N(\mu), p_N(\mu))$ and $\Delta_N^u(\mu)$, $\Delta_N^p(\mu)$ then takes 0.5 and 17.4 milliseconds, respectively, and is thus roughly 410× faster than direct evaluation of the truth approximation. Detailed computation times, also for Algorithm 2, are given in Table 2.3.

(a) Accuracy of at least 1% (resp., 0.1%) for the RB approximations $u_N(\mu)$

Method	N_Z	N	$(u_N(\mu), p_N(\mu))$	$\Delta_N^u(\mu), \Delta_N^p(\mu)$	Total
Algorithm 1	57 (87)	19 (29)	0.8 (1.5)	24.6 (45.5)	25.4 (47.0)
Algorithm 2	45 (84)	20 (37)	0.6 (1.3)	17.6 (39.3)	18.2 (40.6)
Algorithm 3	43 (65)	17 (27)	0.5 (0.9)	17.4 (27.4)	17.9 (28.3)

(b) Accuracy of at least 1% (resp., 0.1%) for the RB approximations $p_N(\mu)$

Method	N_Z	N	$(u_N(\mu), p_N(\mu))$	$\Delta_N^u(\mu), \Delta_N^p(\mu)$	Total
Algorithm 1	78 (108)	26 (36)	1.2 (2.4)	38.3 (65.4)	39.5 (67.8)
Algorithm 2	65 (102)	29 (45)	0.9 (1.8)	27.1 (53.8)	28.0 (55.6)
Algorithm 3	61 (73)	24 (31)	0.8 (1.0)	25.7 (32.2)	26.5 (33.2)

Table 2.3

Computation times in milliseconds for the Online evaluation of $(u_N(\mu), p_N(\mu))$ (assembly and solution of (2.1)) and the error bounds $\Delta_N^u(\mu)$, $\Delta_N^p(\mu)$ (see (2.11), (2.12)); times are measured using either Algorithm 1, Algorithm 2 with $\delta_{\text{tol}}^\beta = 0.1$, or Algorithm 3 with $\delta_{\text{tol}}^\beta = 0.1$, with a prescribed accuracy of 1% (resp., 0.1%) for the RB approximations (a) $u_N(\mu)$ and (b) $p_N(\mu)$.

3. Conclusion. We present in this paper a new RB approach for saddle point problems. Based on Brezzi's theory [4], we not only derive new rigorous *a posteriori* bounds for the errors in the RB approximations but also present a new option to construct stable RB approximation spaces. Compared to earlier approaches, the resulting methods allow us to consider flow problems in parametrized domains with notably greater ease.

The developed rigorous *a posteriori* RB error bounds exhibit significant advantages over existing RB error estimates based on either Babuška's theory for general noncoercive problems or a penalty approach. First, they do not rely on the highly expensive Offline stage necessary for the efficient Online calculation of lower bounds to the truth Babuška inf-sup constants, but only on much less expensive calculations associated with the truth continuity, coercivity, and Brezzi inf-sup constants. Second, as separate upper bounds for the errors in the approximations of the primal variable and the Lagrange multiplier, they enable the systematic estimation of engineering outputs depending on either of the two. Third, numerical results demonstrate that the bounds provided for the errors in the approximations for the primal variable are reasonably sharp.

Both theoretical and numerical results indicate that an additional enrichment of the RB approximation space with supremizer functions is not necessarily optimal. Adaptively recognizing the need for stabilization, a new sampling procedure for saddle point problems constructs RB approximation spaces that provide accurate approximations at much less computational cost: Depending on the employed enrichment strategy, we obtain for our Stokes model problem savings compared to the standard greedy approach of up to 32%. This promises even more significant savings when extending the method to nonlinear problems such as the Navier–Stokes equations.

Even though the reduced basis method is based on an affine decomposition of the involved operators as in (1.7), problems with non-affine parameter-dependencies may also be handled efficiently: In this case, techniques as proposed in [2, 12] can be applied to recover the setting of §1, making the proposed approach also feasible, e.g., in the context of non-affine geometric transformations (see [7, 30]).

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