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H. Egger and C. Waluga

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A HYBRID MORTAR METHOD FOR INCOMPRESSIBLE FLOW

HERBERT EGGER[1] AND CHRISTIAN WALUGA[2]

ABSTRACT. In this paper, we consider the discretization of the Stokes problem on domain partitions with non-matching meshes. We propose a hybrid mortar method, which is motivated by a variational characterization of solutions of the corresponding interface problem. The discretization of the subdomain problems is based on standard inf-sup stable finite element pairs, and the introduction of additional unknowns at the interface allows to reduce the coupling between subdomains, which comes from the variational incorporation of interface conditions. The discrete inf-sup stability condition is proven under weak assumptions on the interface mesh, and optimal a-priori error estimates are derived with respect to the energy and L^2-norm. The theoretical results are illustrated with numerical tests.

Keywords: Stokes equation, interface problems, discontinuous Galerkin methods, hybridization, mortar methods, non-matching grids

AMS subject classification: 65N30, 65N55

1. INTRODUCTION

Various applications in computational fluid dynamics involve moving geometries, multiple physical phenomena, or discontinuous material properties. As typical examples, let us mention the flow around spinning propellers, fluid-structure interaction, groundwater contaminant transport, or multiphase flows. For such problems, it may be convenient to use independent discretizations for subdomain problems, which can be non-matching across the interfaces; e.g., in the hydrodynamic simulation of rotating propellers it is common practice to generate independent meshes for the rotor and the stator domain. Continuity of the solution is then obtained by imposing appropriate coupling conditions on the cylindrical interface.

Methods that incorporate interface conditions in a variational framework allow to deal with non-matching meshes more or less automatically. A prominent example are the classical mortar methods [13], which enforce jump conditions across the interface by Lagrange multipliers. Mortar methods are well-studied, cf. e.g. [16, 52], but they have certain peculiarities. For instance, the space of Lagrange multipliers has to be chosen with care in order to retain stability on the discrete level, and the resulting linear systems are of saddlepoint type, and therefore require appropriate solvers.

An alternative variational approach for the discretization of interface problems is offered by Nitsche-type mortaring techniques [11]; see also [30, 49, 33]. Such techniques avoid the use of Lagrange multipliers, and consequently, the resulting linear systems are positive definite and can be solved with standard iterative methods like the (preconditioned) conjugate gradient method. A drawback of Nitsche-type

[1]Chair for Numerical Mathematics, Technische Universität München, Boltzmannstraße 3, 85748 Garching, Germany. Email: herbert.egger@ma.tum.de

[2]Aachen Institute for Advanced Study in Computational Engineering Science, RWTH Aachen University, Schinkelstraße 2, 52062 Aachen, Germany. Email: waluga@aices.rwth-aachen.de

mortaring is that a lot of coupling is introduced across the interface. This amounts to the large stencils of discontinuous Galerkin methods, which complicates the independent solution of the subdomain problems, and limits the applicability in domain decomposition algorithms [44, 50]. The strong coupling of the subdomain problems can however be relaxed by *hybridization* [25], i.e., by the introduction of additional unknowns at the interface; see also [4, 21] for mixed problems, and also [10, 20] for interface problems and domain decomposition. The hybrid methods yield again positive definite linear systems, and inherit the great flexibility in the choice of ansatz spaces from the Nitsche-type methods, but without introducing their strong coupling.

The aim of this paper is to extend the theoretical framework of hybrid mortar methods [25, 27] to the Stokes system. We start from a variational characterization for the Stokes interface problem, which serves as the starting point for the construction of the hybrid mortar method. The analysis is presented in detail for a two dimensional model problem, and we then discuss how the results can be generalized in order to cover a variety of finite element discretizations in two or three dimensions. Stability of the discrete problems is obtained under mild conditions on the domain partition; in particular, the meshes on the subdomains can be chosen almost completely independent from each other.

Let us mention some further related work: The discretization of Stokes interface problems was investigated in the context of classical mortar methods for instance in [12, 29], and in the framework of discontinuous Galerkin methods in [33, 49, 30]. Hybridization has also been used for the formulation of discontinuous Galerkin methods for Stokes flow [41], and the analysis of the vorticity formulation of Stokes' problem [26]. The approach discussed in this paper however differs in the type of application or discretization. Other aspects of interface problems for Stokes flow, e.g. the use in domain decomposition algorithms, are discussed in [44, 50]; see also [42] for estimates of the inf-sup stability constants independent of jumps in the viscosity.

The plan for our presentation is as follows: In Section 2, we state the Stokes interface problem and derive a variational characterization of solutions to this problem based on a three-field formulation [10, 20]. This characterization is the starting point for the formulation of a hybrid mortar finite element method, and in Sections 3 and 4, we present in detail the stability and error analysis for a specific discretization of a two dimensional model problem. Section 5 then discusses the generalization of the results to three dimensions and more general inf-sup stable finite element spaces. Section 6 finally presents some numerical tests in support of the theoretical results.

2. An interface problem for Stokes flow

2.1. **The Stokes problem.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain in $d = 2, 3$ space dimensions. As a model for the flow of an incompressible viscous fluid confined in Ω, we consider the stationary Stokes problem with homogeneous Dirichlet boundary conditions, i.e.,

$$\text{(1)} \qquad \left\{ \begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned} \right.$$

In order to guarantee uniqueness of the pressure p, we assume as usual that the pressure has zero average. The solutions of the Stokes problem are characterized by the mixed variational principle: Find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$, such that

$$
\text{(2)} \qquad \left\{ \begin{array}{lcllll} a(\boldsymbol{u},\boldsymbol{v}) & + & b(v,p) & = & (f,v) & \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \\ b(\boldsymbol{u},q) & & & = & 0 & \text{for all } q \in L_0^2(\Omega), \end{array} \right.
$$

with bilinear forms a and b defined by

$$
a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v}\, dx \qquad \text{and} \qquad b(\boldsymbol{v},q) = -\int_\Omega \operatorname{div} \boldsymbol{v}\, q\, dx.
$$

Here, $A : B = \sum_{i,j} A_{ij} B_{ij}$ denotes the Frobenius inner product of matrices. The natural function spaces appearing in the variational principle are

$$
H_0^1(\Omega) := \big\{ v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega \big\} \quad \text{and} \quad L_0^2(\Omega) := \Big\{ q \in L^2(\Omega) : \int_\Omega q\, dx = 0 \Big\},
$$

and bold letters are used to denote vector valued functions and spaces of such functions. $L^2(\Omega)$ and $H^s(\Omega)$ are the usual Lebesgue and Sobolev spaces [1].

The following two stability conditions guarantee the unique solvability of the variational problem (2) for any right hand side $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$; cf. [19, 21]: There exist positive constants α_Ω and β_Ω depending only on the domain Ω such that

$$
\text{(3)} \qquad a(\boldsymbol{u},\boldsymbol{u}) \geq \alpha_\Omega \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \qquad \text{for all } \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega),
$$

$$
\text{(4)} \qquad \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \geq \beta_\Omega \|q\|_{L^2(\Omega)} \qquad \text{for all } q \in L_0^2(\Omega).
$$

Terms in the denominator are always assumed not to vanish. The ellipticity of the Laplace operator (3) is a direct consequence of the definition of the bilinear form a and the Poincare-Friedrichs inequality [1]. A proof of the surjectivity of the divergence operator, which amounts to condition (4), can be found in [40, 32, 48]; see also [19, 21] for the general theory on mixed variational problems.

For later reference, let us recall some regularity results for the Stokes problem:

Remark 2.1. On domains with smooth (e.g. $C^{1,1}$) boundary, the solution of (1) satisfies $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $p \in H^1(\Omega)$, whenever $f \in \boldsymbol{L}^2(\Omega)$. The same regularity holds for convex polygonal domains in $\mathbb{R}^2$. For the derivation of these and further regularity results in L^p spaces, see e.g. [34, 2] and the references given there.

2.2. **Domain partition and broken Sobolev spaces.** Let us consider a partition $\Omega^N := \{\Omega_i : i = 1, \ldots, N\}$ of the domain Ω into disjoint Lipschitz subdomains Ω_i. By $\partial\Omega^N := \{\partial\Omega_i : i = 1, \ldots, N\}$, we denote the collection of subdomain boundaries $\partial\Omega_i$ with unit normal vectors $\boldsymbol{n}_i$ pointing to the exterior of Ω_i.

To each interface $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $i < j$, between adjacent subdomains, we associate a unique normal vector by $\boldsymbol{n}_{ij} := \boldsymbol{n}_i = -\boldsymbol{n}_j$. The union of all interfaces $\Gamma := \bigcup_{i<j} \Gamma_{ij}$ is called the *skeleton* or *interface* of the partition Ω^N. Note that the collection $\Gamma^N := \{\Gamma_{ij} : 1 \leq i < j \leq N\}$ of all subdomain interfaces yields a natural partition of the skeleton; see Figure 1 for illustration of a typical partition we have in mind.

The restriction of a function $v \in L^2(\Omega)$ to a subdomain Ω_i is denoted by $v_i := v|_{\Omega_i}$. For $s \geq 0$, we then define the broken Sobolev space

$$
H^s(\Omega^N) := \{v \in L^2(\Omega) : v_i \in H^s(\Omega_i) \text{ for all } \Omega_i \in \Omega^N\} \simeq \prod\nolimits_i H^s(\Omega_i).
$$

By $\simeq$ we mean that the two spaces can be identified naturally. The broken Sobolev spaces are equipped with the product topology

$$
(u,v)_{H^s(\Omega^N)} := \sum\nolimits_i (u_i,v_i)_{H^s(\Omega_i)} \qquad \text{and} \qquad \|u\|_{H^s(\Omega^N)} = \sqrt{(u,u)_{H^s(\Omega^N)}}.
$$

Instead of $(u,v)_{H^s(\Omega^N)}$ we also write $(u,v)_{s,\Omega^N}$, and for $s=0$, we also write $(u,v)_\Omega$ or $(u,v)_{\Omega^N}$, in order to stress the fact that the functions u and v are only piecewise defined. By definition, functions in $v \in H^1(\Omega^N)$ have a well-defined piecewise gradient, which is (again) denoted by $\nabla v|_{\Omega_i} = \nabla v_i$.

By $H^s(\partial\Omega^N) = \prod_i H^s(\partial\Omega_i)$, we denote broken Sobolev spaces of functions defined on the boundaries of the subdomains. Note that such functions formally have two values on the interfaces Γ_{ij}. For $s>0$ and $s \neq$ integer, the functions in $H^s(\partial\Omega^N)$ are just the traces of functions in $H^{s+1/2}(\Omega^N)$. Functions in $L^2(\Gamma)$ are identified with functions in $L^2(\partial\Omega^N)$ by doubling their values on the interfaces and extension by zero on $\partial\Omega$. The L^2 scalar products of functions supported on the skeleton or the domain boundaries are denoted by $\langle\cdot,\cdot\rangle_{\Gamma^N}$ and $\langle\cdot,\cdot\rangle_{\partial\Omega^N}$, respectively, and the corresponding norms are denoted by $|\cdot|_{\Gamma^N}$ and $|\cdot|_{\partial\Omega^N}$.

For a variational formulation of the interface problem, we particularly require

$$H^{1/2}_{00}(\Gamma) := \{v|_\Gamma : v \in H^1_0(\Omega)\}, \qquad \text{and}$$
$$H^{-1/2}(\partial\Omega^N) := \{\boldsymbol{\sigma}\cdot\boldsymbol{n}|_{\partial\Omega^N} : \boldsymbol{\sigma} \in \boldsymbol{H}(\operatorname{div};\Omega^N)\}.$$

Here, $\boldsymbol{H}(\operatorname{div},\Omega^N) \simeq \prod_i \boldsymbol{H}(\operatorname{div};\Omega_i)$ is the space of functions with piecewise well-defined divergence in $L^2(\Omega_i)$.

All definitions naturally extend to vector and tensor valued functions and spaces of such functions, which are denoted with bold symbols.

2.3. **The interface problem.** Under the assumption that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the Stokes problem (1) can be shown to be equivalent to the following interface problem [44]:

$$\begin{cases} -\Delta\,\boldsymbol{u}_i + \boldsymbol{\nabla} p_i &= \boldsymbol{f}_i \qquad \text{in } \Omega_i,\\ \operatorname{div}\boldsymbol{u}_i &= 0 \qquad \text{in } \Omega_i,\\ \boldsymbol{u}_i &= \boldsymbol{0} \qquad \text{on } \partial\Omega\cap\partial\Omega_i,\\ \boldsymbol{u}_i &= \boldsymbol{u}_j \qquad \text{on } \Gamma_{ij},\\ \boldsymbol{T}_i\boldsymbol{n}_{ij} &= \boldsymbol{T}_j\boldsymbol{n}_{ij} \qquad \text{on } \Gamma_{ij}, \end{cases} \tag{5}$$

for all $1 \le i,j \le N$ and $i<j$. Here $\boldsymbol{T} := -\boldsymbol{\nabla}\boldsymbol{u} + p\boldsymbol{I}$ is the stress tensor associated with the Stokes problem.

The two sets of interface conditions ensure the continuity of the velocity field $\boldsymbol{u}$ and the normal stresses $\boldsymbol{T}\boldsymbol{n}_{ij}$ across the interfaces Γ_{ij}, which imply conservation of mass and momentum. The continuity condition on the normal stresses has to be understood in a weak form; cf. [21, Ch.III] and Section 2.4 below. For uniqueness of the pressure, we again require that $\sum_i \int_{\Omega_i} p_i\,dx = 0$ holds.

2.4. **A variational principle for the interface problem.** Different weak formulations for the interface problem (5) can be derived on the continuous or the discrete level; cf. e.g. [44, 12]. As a motivation for the finite element method discussed in this paper, we will utilize the following generalization of the three-field formulation of Brezzi et. al. [20, 44], to the Stokes interface problem; see also [10] for result similar to the following statement.

Lemma 2.2. *Given data $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, let $(\boldsymbol{u},p)$ denote the solution of the Stokes problem* (2) *or the interface problem* (5)*, respectively. Then $\hat{\boldsymbol{u}} := \boldsymbol{u}|_\Gamma \in H^{1/2}_{00}(\Gamma)$ and $\boldsymbol{\lambda} := \boldsymbol{T}\boldsymbol{n}|_{\partial\Omega^N} = \boldsymbol{\partial}_n\boldsymbol{u} - p\boldsymbol{n}|_{\partial\Omega^N} \in \boldsymbol{H}^{-1/2}(\partial\Omega^N)$, and there holds*

$$\begin{cases} (\boldsymbol{\nabla}\boldsymbol{u},\boldsymbol{\nabla}\boldsymbol{v})_{\Omega^N} - (p,\operatorname{div}\boldsymbol{v})_{\Omega^N} - \langle\boldsymbol{\lambda},\boldsymbol{v}-\hat{\boldsymbol{v}}\rangle_{\partial\Omega^N} &= (\boldsymbol{f},\boldsymbol{v})_{\Omega^N}\\ -(\operatorname{div}\boldsymbol{u},q)_{\Omega^N} &= 0\\ -\langle\boldsymbol{u}-\hat{\boldsymbol{u}},\boldsymbol{\mu}\rangle_{\partial\Omega^N} &= 0 \end{cases} \tag{6}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega^N)$, $q \in L^2_0(\Omega)$, $\hat{\boldsymbol{v}} \in \boldsymbol{H}^{1/2}_{00}(\Gamma)$, *and* $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\partial\Omega^N)$.

Conversely, the variational principle (6) *has a unique solution* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega^N)$, $p \in L^2_0(\Omega)$, $\hat{\boldsymbol{u}} \in \boldsymbol{H}^{1/2}_{00}(\Gamma)$, *and* $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\partial\Omega^N)$, *and* $(\boldsymbol{u}, p)$ *which coincides with the solution of the Stokes problem.*

Proof. The statement is a generalization of [20, Thm 1] to the Stokes problem; see also [10]. If $(\boldsymbol{u}, p)$ is the solution of (1), then $\boldsymbol{T} := -\boldsymbol{\nabla}\boldsymbol{u} + p\boldsymbol{I} \in \boldsymbol{H}(\mathrm{div}\,;\Omega)$, since $\mathrm{div}\,\boldsymbol{T} = \boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. It follows that $\boldsymbol{T}_i \in \boldsymbol{H}(\mathrm{div}\,;\Omega_i)$, and thus $\boldsymbol{T}_i\boldsymbol{n}_i \in \boldsymbol{H}^{-1/2}(\partial\Omega_i)$ is well-defined; in our notation, we write $\boldsymbol{T} \in \boldsymbol{H}(\mathrm{div}\,;\Omega^N)$ and $\boldsymbol{T}\boldsymbol{n} \in \boldsymbol{H}^{-1/2}(\partial\Omega^N)$. By the Gauß-Green formula, and summation over all subdomains, we then have

$$(\boldsymbol{f}, \boldsymbol{v})_{\Omega^N} = (\mathrm{div}\,\boldsymbol{T}, \boldsymbol{v})_{\Omega^N} = (\boldsymbol{\nabla}\boldsymbol{u} - p\boldsymbol{I}, \boldsymbol{\nabla}\boldsymbol{v})_{\Omega^N} - \langle \boldsymbol{\partial}_n\boldsymbol{u} - p\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial\Omega^N},$$

which, together with the definition of $\boldsymbol{\lambda}$, yields the first equation in (6) for $\hat{\boldsymbol{v}} = 0$. Moreover, we have [21, Prop III.1.2]

$$\langle \boldsymbol{T}\boldsymbol{n}, \hat{\boldsymbol{v}}\rangle_{\partial\Omega^N} = \sum\nolimits_i \langle \boldsymbol{T}_i\boldsymbol{n}_i, \hat{\boldsymbol{v}}\rangle_{\partial\Omega_i} = 0 \qquad \text{for all } \hat{\boldsymbol{v}} \in \boldsymbol{H}^{1/2}_{00}(\Gamma),$$

which is the proper weak formulation of the interface conditions for the normal stresses. The second equation of (6) follows directly from the incompressibility condition, and the third equation results from $\boldsymbol{u}_i = \boldsymbol{u}_j = \hat{\boldsymbol{u}}$ on Γ_{ij} by definition.

Let us now turn to the converse statement: From the third equation in (6), we obtain that $\boldsymbol{u} = \hat{\boldsymbol{u}}$ on Γ and thus $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$. Testing the first equation with arbitrary $\hat{\boldsymbol{v}}$, it follows that the normal stresses are continuous (single valued) across the interface Γ, and by definition of $\boldsymbol{H}^{1/2}_{00}(\Gamma)$ we obtain that $\langle \boldsymbol{\lambda}, \boldsymbol{v}\rangle_{\partial\Omega^N} = 0$ for all $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$. Using this relation, we obtain from (6) by testing with arbitrary $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$ and $q \in L^2_0(\Omega)$, that $(\boldsymbol{u}, p)$ solves (2). The identity $\boldsymbol{\lambda} = \boldsymbol{\partial}_n\boldsymbol{u} - p\boldsymbol{I}$ now follows from the first equation and the Gauß-Green formula, and uniqueness is obtained from the unique solvability of the Stokes problem (2). □

By explicitly eliminating the normal stresses $\boldsymbol{\lambda}$ from (6), and dropping the third equation, we immediately obtain the following statement, which will be the basis for deriving a consistent variational principle below.

Corollary 2.3. *Let* $(\boldsymbol{u}, p)$ *denote the solution of the Stokes problem* (1) *respectively the interface problem* (5). *Then*

$$\begin{aligned}(\boldsymbol{f}, \boldsymbol{v})_{\Omega^N} &= (\boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\nabla}\boldsymbol{v})_{\Omega^N} - (p, \mathrm{div}\,\boldsymbol{v})_{\Omega^N} - \langle \boldsymbol{\partial}_n\boldsymbol{u} - p\boldsymbol{n}, \boldsymbol{v} - \hat{\boldsymbol{v}}\rangle_{\partial\Omega^N},\\ 0 &= -(\mathrm{div}\,\boldsymbol{u}, q)_{\Omega^N},\end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega^N)$, $\hat{\boldsymbol{v}} \in \boldsymbol{H}^{1/2}_{00}(\Gamma)$, *and* $q \in L^2_0(\Omega)$.

Remark 2.4. As above, we can define $\hat{\boldsymbol{u}} := \boldsymbol{u}|_\Gamma$ and then deliberately add interface terms of the form $\langle \boldsymbol{u} - \hat{\boldsymbol{u}}, \cdot\rangle_{\partial\Omega^N}$ without changing the validity of the variational principle. We will make use of such additional terms for devising stable discretization schemes in the next section.

3. Discretization by finite elements

For ease of presentation, we consider in the following the two dimensional problem and a simple choice of finite element spaces. The application of the proposed method to a more general setting will be clear from the exposition, and some generalizations are discussed in some detail in Section 5.

3.1. **The finite element mesh.** Let $\Omega \subset \mathbb{R}^2$ be partitioned into non-overlapping polygonal subdomains Ω_i, and let $\mathcal{T}_h(\Omega_i)$ denote conforming triangulations of the subdomains Ω_i into elements T. We call $\mathcal{T}_h(\Omega^N) := \bigcup_i \mathcal{T}_h(\Omega_i)$ the global mesh, and we assume that $\mathcal{T}_h(\Omega^N)$ is γ-shape-regular, i.e.,

$$\rho_T/h_T \geq \gamma > 0 \qquad \text{for all } T \in \mathcal{T}_h(\Omega^N).$$

As usual, $h_T := \mathrm{diam}(T)$ is the size of the element T, i.e., the local meshsize, and ρ_T denotes the diameter and largest ball inscribed in T.

We further require conforming partitions $\mathcal{E}_h(\Gamma_{ij})$ of the interfaces $\Gamma_{ij} \subset \Gamma^N$ into segments E of size h_E, and we denote by $\mathcal{E}_h(\Gamma^N) := \bigcup_{i<j} \mathcal{E}_h(\Gamma_{ij})$ the total mesh of the interface.

We assume that the mesh $(\mathcal{T}_h, \mathcal{E}_h)$ is locally quasi-uniform, i.e., neighbouring elements are of comparable size, and we assume that

$$h_T \leq \gamma^{-1} h_{T'} \quad \text{or} \quad h_E \leq \gamma^{-1} h_T, \qquad \text{if} \quad T \cap T' \neq \emptyset \text{ or } \partial T \cap E \neq \emptyset. \tag{7}$$

The typical situation is depicted in Figure 1.

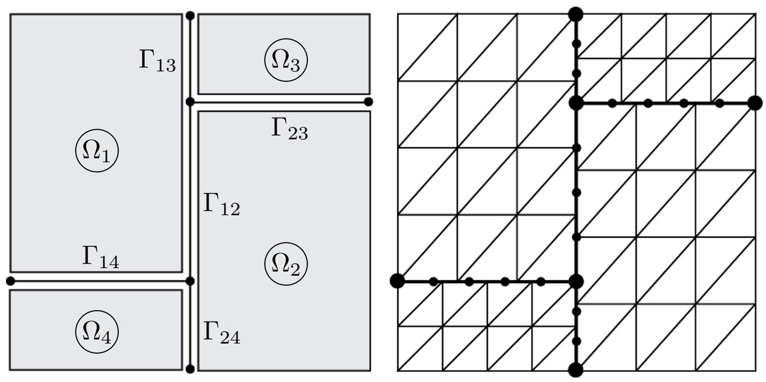

FIGURE 1. Domain partition $\Omega^N = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$ and corresponding partition of the skeleton Γ given by $\Gamma^N = \{\Gamma_{12}, \Gamma_{13}, \Gamma_{14}, \Gamma_{23}, \Gamma_{24}\}$ (left). A possible triangulation used for the first test problem in Section 6 is depicted on the right. The crosspoints of the domain partition are depicted with (bold) circles.

Remark 3.1. We implicitly assumed that the interface mesh $\mathcal{E}_h$ is "conforming", in the sense that the cross-points of the domain decomposition are vertices of the interface meshes. Similar conditions are employed for the analysis of discontinuous Galerkin discretizations in [49, 33]. Also note that (7) implies that the size $|\Gamma_{ij}|$ of the interfaces is bounded from below by the local meshsize h_E respectively h_T.

3.2. **A hybrid mortar finite element method.** For approximating the solutions of the Stokes (interface) problem, we consider piecewise polynomial function spaces made up of inf-sup stable elements on the subdomains. In the sequel, we consider in detail the choice

$$\boldsymbol{V}_h := \left\{ \boldsymbol{v}_h \in \boldsymbol{H}^1(\Omega^N) : \boldsymbol{v}_h|_T \in [\mathcal{P}_2(T)]^2 \quad \text{for all} \quad T \in \mathcal{T}_h(\Omega^N) \right\},$$
$$Q_h := \left\{ q_h \in L^2_0(\Omega) : q_h|_T \in \mathcal{P}_0(T) \quad \text{for all} \quad T \in \mathcal{T}_h(\Omega^N) \right\},$$

which is a natural extension of the well-known $\mathcal{P}_2 - \mathcal{P}_0$ element to the variational setting of the interface problem introduced in the previous section. As usual, $\mathcal{P}_k(T)$ denotes the space of polynomials of maximal order k on the element T.

Remark 3.2. By construction, the restriction of the discrete spaces to the subdomains yields inf-sup stable finite element pairs for the subdomain problems. This will be a main ingredient for the stability analysis of the next section. Other stable pairs could however be chosen as well; see Section 5 for details.

For approximation of the velocities on the skeleton, we then utilize a space of discontinuous piecewise quadratic functions, namely

$$\widehat{\boldsymbol{V}}_h := \left\{ \hat{\boldsymbol{v}}_h \in \boldsymbol{L}^2(\Gamma^N) : \hat{\boldsymbol{v}}_h|_E \in [\mathcal{P}_2(E)]^2 \quad \text{for all} \quad E \in \mathcal{E}_h(\Gamma^N) \right\}.$$

The polynomial order of the interface space is chosen in order to have similar approximation properties as the trace of the space $\boldsymbol{V}_h$.

Remark 3.3. Since the space $\widehat{\boldsymbol{V}}_h$ is not a subspace of the test space $\boldsymbol{H}^{1/2}_{00}(\Gamma)$ used in the variational characterization of Lemma 2.2 or Corollary 2.3, the proposed spaces will lead to non-conforming approximations. In principle also a conforming space could be chosen here, cf. [10], but we will make explicit use of the non-conformity in the proof of the discrete inf-sup stability below.

For discretization of the interface problem, we then consider the following method.

Method 3.1 (Hybrid mortar method)**.** *Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, find $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{u}}_h \in \widehat{\boldsymbol{V}}_h$, and $p_h \in Q_h$, such that*

$$\begin{aligned} a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) + b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p_h) &= (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega, \qquad && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h, \\ b_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; q_h) &= 0, && \forall q_h \in Q_h, \end{aligned}$$

with bilinear forms a_h and b_h defined by

$$\begin{aligned} a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) &:= (\boldsymbol{\nabla} \boldsymbol{u}, \boldsymbol{\nabla} \boldsymbol{v})_{\Omega^N} - (p, \operatorname{div} \boldsymbol{v})_{\Omega^N} - \langle \boldsymbol{\partial}_n \boldsymbol{u} - p\boldsymbol{n}, \boldsymbol{v} - \hat{\boldsymbol{v}} \rangle_{\partial\Omega^N} \\ &\qquad - \langle \boldsymbol{u} - \hat{\boldsymbol{u}}, \boldsymbol{\partial}_n \boldsymbol{v} \rangle_{\partial\Omega^N} + \langle \alpha(\boldsymbol{u} - \hat{\boldsymbol{u}}), \boldsymbol{v} - \hat{\boldsymbol{v}} \rangle_{\partial\Omega^N}, \\ b(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; q_h) &:= -(\operatorname{div} \boldsymbol{u}, q)_{\Omega^N} + \langle q\boldsymbol{n}, \boldsymbol{u} - \hat{\boldsymbol{u}} \rangle_{\partial\Omega^N}. \end{aligned}$$

The elementwise constant stabilization parameter $\alpha > 0$ will be specified below.

3.3. **Relation to other methods.** Before we turn to the analysis of Method 3.1, let us make some remarks concerning the relation to other methods:

Method 3.1 can be interpreted as particular realization of a three-field method [10, 20] with appropriate stabilization terms. This was also our starting point for the derivation of the method. In contrast to [10, 20], we however consider a non-conforming (discontinuous) discretization of the hybrid variable here, which facilitates the verification of the discrete inf-sup stability.

The hybrid mortar method is dual to the classical mortar methods [13, 16, 12], where, instead of the trace $\hat{\boldsymbol{u}}$, the normal stress $\boldsymbol{\lambda}$ (see Lemma 2.2) appears as additional variable. For a short discussion on the relation of classical mortar methods to the three-field formulation see [44].

Explicit elimination of the hybrid variable yields a discontinuous Galerkin method similar to those discussed in [33, 30]. Keeping the hybrid variable in the formulation however simplifies the analysis and implementation of the method, and facilitates the application of domain decomposition algorithms [44, 50].

Other hybrid formulations of the Stokes and Navier-Stokes problem have been proposed and analyzed in [41, 38]; see also [33, 30, 46] for related discontinuous Galerkin methods. For a general approach to the hybridization of discontinuous Galerkin methods and applications to mortaring, we refer to [25], and to [10, 20] for corresponding three-field formulations.

3.4. **Basic properties of the hybrid mortar method.** The following statement is a direct consequence of Corollary 2.3. We require some extra regularity to make all terms well-defined.

Lemma 3.4 (Consistency)**.** *Assume that the solution of the Stokes problem* (1) *is regular, i.e.,* $\boldsymbol{u} \in \boldsymbol{H}^2(\mathcal{T}_h)$ *and* $p \in H^1(\mathcal{T}_h)$*, and define* $\hat{\boldsymbol{u}} := \boldsymbol{u}|_\Gamma$*. Then*

$$a_h(\boldsymbol{u}, \hat{\boldsymbol{u}}; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) + b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p) = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega,$$
$$b_h(\boldsymbol{u}, \hat{\boldsymbol{u}}; q_h) = 0,$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$*,* $\hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$*, and* $q_h \in Q_h$*, i.e., the hybrid mortar method is consistent.*

Remark 3.5. For conditions on the domain Ω providing sufficient regularity of the solution $(\boldsymbol{u}, p)$, see Remark 2.1. The regularity requirements on the solution can be relaxed in various ways, e.g., by redefining the bilinear forms a_h and b_h, such that $\tilde{a}_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) = a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)$ and $\tilde{b}_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h, q_h) = b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; q_h)$ on the discrete spaces, but such that additionally, $\tilde{a}_h$ and $\tilde{b}_h$ are well-defined and continuous for arguments $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$, and $p \in L_0^2(\Omega)$. For details, let us refer to [43, 37].

Due to the non-conformity of the mesh across the domain interfaces, and the use of H^1 conforming discretizations on the subdomains, the hybrid mortar method is not *locally conservative* in the sense of [5]. However, the following semi-local conservation property holds.

Lemma 3.6. *For every interface* Γ_{ij} *define the* numerical flux $\boldsymbol{\sigma}_k \boldsymbol{n}_{ij}$ *by*

$$\boldsymbol{\sigma}_k \boldsymbol{n}_{ij} := \partial_{\boldsymbol{n}_{ij}} \boldsymbol{u}_{k,h} - p_{k,h} \boldsymbol{n}_{ij} - \alpha(\boldsymbol{u}_{k,h} - \hat{\boldsymbol{u}}_{ij,h}), \qquad k \in \{i, j\}.$$

Then there holds

$$\int_{\Gamma_{ij}} \boldsymbol{\sigma}_i \boldsymbol{n}_{ij} \cdot \mathbf{1}\, ds = \int_{\Gamma_{ij}} \boldsymbol{\sigma}_j \boldsymbol{n}_{ij} \cdot \mathbf{1}\, ds,$$

i.e., the total flux is conserved across the domain interfaces.

Remark 3.7. The conservation property follows immediately from the fact, that the function $\hat{\boldsymbol{v}}_h := \mathbf{1}|_{\Gamma_{ij}}$ is in the test space $\widehat{\boldsymbol{V}}_h$. If the interface mesh is fine enough, i.e., if for any $T \in \mathcal{T}_h(\Omega_i)$, $T' \in \mathcal{T}_h(\Omega_j)$ the intersection $\partial T \cap \partial T'$ can be represented by a union of elements in $\mathcal{E}_h(\Gamma_{ij})$, then conservation also holds locally (on the element level), i.e., the numerical flux is continuous over the interface mesh. This follows directly from the discrete variational principle by testing with $\hat{\boldsymbol{v}}_h = \mathbf{1}|_E$.

4. Error analysis

4.1. **Preliminaries.** In the subsequent analysis, we will frequently use the following well-known discrete trace inequalities.

Lemma 4.1. *Let* $v \in \mathcal{P}_k(T)$*. Then there exists a constant* C_T *depending only on the shape of the element* T *and the polynomial degree* k*, such that*

$$|v|_{\partial T}^2 \le C_T h_T^{-1} \|v\|_T^2 \qquad \text{and} \qquad |\boldsymbol{\partial}_n v|_{\partial T}^2 \le C_T h_T^{-1} \|\nabla v\|_T^2. \tag{8}$$

Remark 4.2. In general, one has $C_T = c_T k_T^2$, with c_T independent of the polynomial degree. For simple elements (simplices, hypercubes), sharp estimates for c_T in dependence of the shape of the element are available, cf. e.g. [51]. Since we assume uniform shape regularity, we can replace c_T by c_γ depending only on the shape regularity constant γ of the mesh.

We now choose the elementwise constant stabilization parameter α such that

$$\alpha|_T \geq 4C_T h_T^{-1} \qquad \text{and} \qquad \alpha|_T \leq \tilde{C}_T h_T^{-1} \tag{9}$$

for some $\tilde{C}_T \sim C_T$. Next, we define a pair of discrete trace norms

$$|v|_{1/2,h} := |\alpha^{\frac{1}{2}} v|_{\partial\Omega^N} \quad \text{and} \quad |v|_{-1/2,h} := |\alpha^{-\frac{1}{2}} v|_{\partial\Omega^N}. \tag{10}$$

Such pairs of discrete trace norms are commonly used for the analysis of non-conforming finite element methods; cf. e.g. [3, 16, 5]. Note that by definition of the norms and the Cauchy-Schwarz inequality, we have

$$\langle u, v\rangle_{\partial\Omega^N} \leq |u|_{1/2,h} |v|_{-1/2,h} \qquad \text{for all } u, v \in L^2(\partial\Omega^N), \tag{11}$$

which mimicks the usual estimate for the duality pairing.

As an immediate consequence of the definition of the discrete trace norms, and the assumptions on the mesh and the polynomial spaces, we obtain the following:

Lemma 4.3. *Let α be chosen as in* (9)*. Then there holds*

$$|\boldsymbol{\partial}_n \boldsymbol{v}_h|_{-1/2,h} \leq \tfrac{1}{2} \|\boldsymbol{\nabla} \boldsymbol{v}_h\|_{\Omega^N} \qquad \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{12}$$

In the following sections, we will establish the conditions needed for the application of Brezzi's theorem [21], which guarantees the existence and uniqueness of a finite element solution for the discrete variational problem. For our stability analysis, we will utilize the energy norms

$$\|(\boldsymbol{v}, \hat{\boldsymbol{v}})\|_{1,h} := \big(\|\boldsymbol{\nabla} \boldsymbol{v}\|^2_{\Omega^N} + |\boldsymbol{v} - \hat{\boldsymbol{v}}|^2_{1/2,h}\big)^{1/2} \qquad \text{and} \qquad \|q\|_{0,h} := \|q\|_{\Omega^N}.$$

Similar mesh-dependent norms are frequently used in the analysis of discontinuous Galerkin finite element methods, cf. e.g., [3, 5].

4.2. **Coercivity and boundedness.** Let us start with stating the coercivity and boundedness of the bilinear forms on the finite element spaces.

Proposition 4.4 (Ellipticity)**.** *Let the stabilization parameter α be chosen according to* (9)*. Then for all $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h) \in \boldsymbol{V}_h \times \widehat{\boldsymbol{V}}_h$ there holds*

$$a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{u}_h, \hat{\boldsymbol{u}}_h) \geq \tfrac{1}{2} \|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|^2_{1,h}. \tag{13}$$

Proof. The result follows directly from the discrete trace inequality (8), and the Cauchy-Schwarz and Young's inequalities. □

Proposition 4.5 (Boundedness)**.** *For all functions $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$, and $p_h, q_h \in Q_h$, there holds*

$$a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{u}_h, \hat{\boldsymbol{v}}_h) \leq C_a \|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} \|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h},$$
$$b_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; p_h) \leq C_b \|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} \|p_h\|_{0,h},$$

with constants $C_a = 3/2$, $C_b = \sqrt{5/4}$ independent of the meshsize h.

Proof. The result follows again directly from the Cauchy-Schwarz inequality, the discrete trace inequality (8), and the definition of the norms. □

4.3. **Inf-sup stability.** For establishing the discrete inf-sup stability condition for the bilinear form b_h, we require some additional results: A basic ingredient for our analysis is that the restriction of the spaces V_h and Q_h to the subdomains, i.e.,

$$\boldsymbol{V}_h(\Omega_i) := \boldsymbol{V}_h|_{\Omega_i} \cap \boldsymbol{H}_0^1(\Omega_i) \qquad \text{and} \qquad Q_h(\Omega_i) := Q_h|_{\Omega_i} \cap L_0^2(\Omega_i) \tag{14}$$

are inf-sup stable finite element pairs for the Stokes problem on the subdomain Ω_i. For later reference, let us recall the discrete inf-sup stability of the $\mathcal{P}_2 - \mathcal{P}_0$ element, cf. e.g. [32, 14], which serves as discrete analogon of (4).

Lemma 4.6. *The spaces $\boldsymbol{V}_h(\Omega_i)$, $Q_h(\Omega_i)$ satisfy the discrete inf-sup condition*

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega_i)} \frac{b_i(\boldsymbol{v}_h, p_h)}{\|\boldsymbol{v}_h\|_{1,\Omega_i}} \geq \beta_i \|p_h\|_{0,\Omega_i} \qquad \forall\, p_h \in Q_h(\Omega_i), \tag{15}$$

with some $\beta_i > 0$ independent of the meshsize h.

We further require some results on projection and quasi-interpolation operators.

Lemma 4.7 (Quasi-interpolation)**.** *There exists linear bounded projection operator $\mathcal{C}_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h$, such that for all functions $\boldsymbol{v} \in \boldsymbol{H}_0^1$ there holds*

$$\|\nabla \mathcal{C}_h \boldsymbol{v}\|_{\Omega^N} \leq C' \|\nabla \boldsymbol{v}\|_{\Omega^N} \qquad \text{and} \qquad \Big(\sum\nolimits_T h_T^{-2} \|\mathcal{C}_h \boldsymbol{v} - \boldsymbol{v}\|_T^2\Big)^{1/2} \leq C' \|\nabla \boldsymbol{v}\|^2\Big)^{1/2}$$

with a constant C' independent of the meshsize.

Proof. Since the space $\boldsymbol{V}_h$ does not require continuity across subdomain interfaces, the operator $\mathcal{C}_h$ can be defined subdomain wise by $(\mathcal{C}_h \boldsymbol{v})|_{\Omega_i} := \mathcal{C}_i(\boldsymbol{v}|_{\Omega_i})$, where $\mathcal{C}_i$ is a quasi-interpolation operator for the subdomain Ω_i, with the required properties; cf. [23, 47]. The result then follows by summation over the subdomains. □

Lemma 4.8. *Let $\widehat{\boldsymbol{\Pi}}_h : \boldsymbol{H}_0^1(\Omega) \to \widehat{\boldsymbol{V}}_h$ be the orthogonal projector with respect to $L^2(\Gamma)$, and $\mathcal{C}_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h$ be defined as in Lemma 4.7. Then*

$$\|(\mathcal{C}_h \boldsymbol{v}, \widehat{\boldsymbol{\Pi}}_h \boldsymbol{v})\|_{1,h} \leq C'' \|\nabla \boldsymbol{v}\|_\Omega$$

for all functions $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ with a constant C'' independent of the meshsize. Morover, there holds

$$\langle \widehat{\boldsymbol{\Pi}}_h \boldsymbol{v}, \boldsymbol{n} \rangle_{\Gamma_{ij}} = \langle \boldsymbol{v}, \boldsymbol{n} \rangle_{\Gamma_{ij}} \qquad \text{for all } \Gamma_{ij} \in \Gamma^N.$$

Proof. For convenience, let us denote $\mathcal{C}_h \boldsymbol{v} =: \boldsymbol{v}_h$ and $\widehat{\boldsymbol{\Pi}}_h \boldsymbol{v} =: \hat{\boldsymbol{v}}_h$. Using the triangle inequality, we split the boundary terms appearing in the norm $\|(\cdot,\cdot)\|_{1,h}$ by

$$|\boldsymbol{v}_h - \hat{\boldsymbol{v}}_h|_{1/2,h} \leq |\boldsymbol{v}_h - \boldsymbol{v}|_{1/2,h} + |\hat{\boldsymbol{v}}_h - \boldsymbol{v}|_{1/2,h}.$$

A continuous trace inequality [3, eq. (2.4)], and quasi-uniformity (7) and shape-regularity of the mesh then yield

$$h_T^{-1} |\boldsymbol{v}_h - \boldsymbol{v}|_{\partial T}^2 \leq c \left(\|\nabla(\boldsymbol{v}_h - \boldsymbol{v})\|_T^2 + h_T^{-2} \|\boldsymbol{v}_h - \boldsymbol{v}\|_T^2 \right),$$

with constant c only depending on the shape of the element T. Summing over all elements, and using the stability and approximation properties of the quasi-interpolation, we obtain

$$|\boldsymbol{v}_h - \boldsymbol{v}|_{1/2,h} \leq C \|\nabla \boldsymbol{v}\|_\Omega,$$

with constant C independent of the meshsize and the function $\boldsymbol{v}$.

From the best approximation property of the L^2 projection and the polynomial approximation results of [8], we further obtain $|\hat{\boldsymbol{v}}_h - \boldsymbol{v}|_{1/2,h} \leq C \|\nabla \boldsymbol{v}\|_\Omega$, which yields the required estimate for the boundary terms of the norm.

The stability of the quasi-interpolation follows, and the condition $h_E \leq \gamma^{-1} h_T$ for adjacent elements yields $\|\nabla \boldsymbol{v}_h\|_{\Omega^N} \leq C' \|\nabla \boldsymbol{v}\|_\Omega$, which completes the proof. □

The following result establishes the discrete inf-sup condition for the hybrid mortar method, which is the essential ingredient for the stability and error analysis.

Theorem 4.9. *The bilinear form b_h satisfies a discrete inf-sup condition, i.e., there exists a constant $\beta > 0$ independent of h such that*

$$\sup_{(\boldsymbol{v}_h,\hat{\boldsymbol{v}}_h)\in \boldsymbol{V}_h\times\widehat{\boldsymbol{V}}_h} \frac{b_h(\boldsymbol{v}_h,\hat{\boldsymbol{v}}_h;q_h)}{\|(\boldsymbol{v}_h,\hat{\boldsymbol{v}}_h)\|_{1,h}} \geq \beta\,\|q_h\|_0, \tag{16}$$

holds uniformly for all functions $q_h \in Q_h$.

Proof. We employ an argument introduced by Boland and Nicolaides [15] to explicitly construct a pair of functions $(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)$ that satisfies the inequality.

Step 1: Any function $q_h \in Q_h$ can be decomposed into

$$q_h = q_h^a + q_h^b,$$

where the function q_h^b is constant on each subdomain, and q_h^a has zero mean on each subdomain, i.e.,

$$q_h^b \in L_0^2(\Omega),\ q_h^b|_{\Omega_i} = \text{const.} \qquad \text{and} \qquad q_h^a|_{\Omega_i} \in L_0^2(\Omega_i).$$

Note that the two functions are orthogonal with respect to $L^2(\Omega^N)$.

Step 2: Due to the discrete inf-sup stability of the pair $\boldsymbol{V}_h(\Omega_i)$, $Q_h(\Omega_i)$ (see Lemma 4.6), we can define a function $\boldsymbol{v}_h^a \in \boldsymbol{V}_h$, such that $\boldsymbol{v}_h^a|_{\Omega_i} \in \boldsymbol{V}_h(\Omega_i)$ on each subdomain, and

$$-(\operatorname{div}\boldsymbol{v}_h^a, q_h^a)_{\Omega_i} = \|q_h^a\|_{\Omega_i}^2 \qquad \text{with} \qquad \|\boldsymbol{v}_h^a\|_{1,\Omega_i} \leq \beta_i^{-1}\,\|q_h^a\|_{\Omega_i}.$$

Setting $\hat{\boldsymbol{v}}_h^a := \boldsymbol{0}$, and summing over all subdomains yields

$$b_h(\boldsymbol{v}_h^a,\hat{\boldsymbol{v}}_h^a;q_h^a) = \|q_h^a\|_{0,h}^2 \qquad \text{and} \qquad \|(\boldsymbol{v}_h^a,\hat{\boldsymbol{v}}_h^a)\|_{1,h} \leq \beta_a^{-1}\|q_h^a\|_{0,h}. \tag{17}$$

with constant β_a defined by $\beta_a^{-1} := \max_i \beta_i^{-1}$.

Step 3: By the surjectivity of the divergence operator on the continuous level (4), there exists a function $\boldsymbol{v}^b \in \boldsymbol{H}_0^1(\Omega)$ such that

$$-\operatorname{div}\boldsymbol{v}^b = q_h^b \qquad \text{and} \qquad \|\boldsymbol{v}^b\|_{1,\Omega} \leq \beta_\Omega^{-1}\|q_h^b\|_{0,h}.$$

Defining $\boldsymbol{v}_h^b := \mathcal{C}_h\boldsymbol{v}^b$ and $\hat{\boldsymbol{v}}_h^b := \widehat{\boldsymbol{\Pi}}_h\boldsymbol{v}^b$, with $\mathcal{C}_h$, $\widehat{\boldsymbol{\Pi}}_h$ as in Lemma 4.8, and integrating by parts, we obtain

$$b_h(\boldsymbol{v}_h^b,\hat{\boldsymbol{v}}_h^b;q_h^b) = (\boldsymbol{v}_h^b,\nabla q_h^b)_{\Omega^N} - \langle\hat{\boldsymbol{v}}_h^b, q_h^b\boldsymbol{n}\rangle_{\partial\Omega^N} = -(\operatorname{div}\boldsymbol{v}^b, q_h^b)_{\Omega^N}.$$

Due to the definition of $\boldsymbol{v}^b$ and Lemma 4.8, we further obtain that

$$b_h(\boldsymbol{v}_h^b,\hat{\boldsymbol{v}}_h^b;q_h^b) = \|q_h^b\|_{0,h}^2 \qquad \text{and} \qquad \|(\boldsymbol{v}_h^b,\hat{\boldsymbol{v}}_h^b)\|_{1,h} \leq \beta_b^{-1}\|q_h^b\|_{0,h}, \tag{18}$$

with constant $\beta_b = \beta_\Omega/C''$ and C'' as in Lemma 4.8.

Step 4: For $\varepsilon > 0$, let us now define $\boldsymbol{v}_h := \boldsymbol{v}_h^a + \varepsilon\boldsymbol{v}_h^b$ and $\hat{\boldsymbol{v}}_h := \hat{\boldsymbol{v}}_h^a + \varepsilon\hat{\boldsymbol{v}}_h^b$. By construction, q_h^b is constant on each subdomain Ω_i. Hence $b_h(\boldsymbol{v}_h^a,\hat{\boldsymbol{v}}_h^a;q_h^b) = 0$, and using (17) and (18), we obtain

$$\begin{aligned} b_h(\boldsymbol{v}_h,\hat{\boldsymbol{v}}_h;q_h) &= b_h(\boldsymbol{v}_h^a,\hat{\boldsymbol{v}}_h^a;q_h^a) + \varepsilon\, b_h(\boldsymbol{v}_h^b,\hat{\boldsymbol{v}}_h^b;q_h^b) + \varepsilon\, b_h(\boldsymbol{v}_h^b,\hat{\boldsymbol{v}}_h^b;q_h^a) \\ &\geq \|q_h^a\|_{0,h}^2 + \varepsilon\,\|q_h^b\|_{0,h}^2 - \varepsilon C_b\|(\boldsymbol{v}_h^b,\hat{\boldsymbol{u}}_h^b)\|_{1,h}\|q_h^a\|_{0,h} \\ &\geq \tfrac{1}{2}\|q_h^a\|_{0,h}^2 + \varepsilon\left(1 - \tfrac{\varepsilon}{2}C_b^2\beta_b^{-2}\right)\|q_h^b\|_{0,h}^2. \end{aligned}$$

Choosing $\varepsilon = \beta_b^2/C_b^2$, and utilizing the orthogonality of q_h^a and q_h^b, we further obtain that $b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; q_h) \geq \frac{1}{2}\min(1, \beta_b^2 C_b^{-2})\|q_h\|_{0,h}^2$. The norm estimates for $\boldsymbol{v}_h^a$ and $\boldsymbol{v}_h^b$ finally imply that

$$\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h} \leq \max(\beta_a^{-1}, \beta_b C_b^{-2})\|q_h\|_{0,h},$$

and (16) follows with $\beta = \frac{1}{2}\min(1, \beta_b^2 C_b^{-2})\min(\beta_a, \beta_b^{-1}C_b^2)$. □

As a direct consequence of the discrete ellipticity and inf-sup stability conditions, we obtain the following stability estimate in the sense of Babuška-Aziz [7].

Theorem 4.10. *For every $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h, p_h) \in \boldsymbol{V}_h \times \widehat{\boldsymbol{V}}_h \times Q_h$ there exists a non-trivial element $(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h, q_h) \in \boldsymbol{V}_h \times \widehat{\boldsymbol{V}}_h \times Q_h$ such that*

$$\begin{aligned} a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) &+ b_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; q_h) + b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p_h) \\ &\geq c_{stab}(\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} + \|p_h\|_{0,h})(\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h} + \|q_h\|_{0,h}), \end{aligned}$$

with constant $c_{stab} = \frac{1}{2}\max\{2 + \beta^{-1}(1 + 2C_a/\beta), \beta^{-1}(1 + 2C_a)(1 + C_a/\beta)\}$.

Proof. Due to the discrete stability conditions and Brezzi's theorem [19, 21], there exists a unique solution $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$, and $q_h \in Q_h$, such that

$$\begin{aligned} a_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; \boldsymbol{w}_h, \hat{\boldsymbol{w}}_h) + b_h(\boldsymbol{w}_h, \hat{\boldsymbol{w}}_h; q_h) &= (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{w}_h)_{0,h} + \langle \alpha(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h), \boldsymbol{w}_h - \hat{\boldsymbol{w}}_h\rangle_{\partial\Omega^N} \\ b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; r_h) &= (p_h, r_h)_{0,h} \end{aligned}$$

for all $\boldsymbol{w}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{w}}_h \in \widehat{\boldsymbol{V}}_h$, and $r_h \in Q_h$. Moreover, there holds

$$\begin{aligned} \|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h} &\leq 2\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} + \beta^{-1}(1 + 2C_a)\|p_h\|_{0,h} \\ \|q_h\|_{0,h} &\leq \beta^{-1}(1 + 2C_a)\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} + \beta^{-1}(1 + 2C_a)(1 + C_a/\beta)\|p_h\|_{0,h}, \end{aligned}$$

which follows from [17, Thm. III.4.3]. Using these test functions and the symmetry of the bilinear form a_h, we obtain

$$\begin{aligned} a_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) &+ b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p_h) + b_h(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h; q_h) \\ &= \|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h}^2 + \|p_h\|_{0,h}^2 \geq \tfrac{1}{2}(\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} + \|p_h\|_{0,h})^2, \end{aligned}$$

which together with the a-priori estimate on $\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h}$ and $\|q_h\|_{0,h}$ yields the assertion. □

An immediate consequence now is the unique solvability of the discrete problem, which yields the well-definedness of the hybrid mortar method.

Theorem 4.11. *Method 3.1 has a unique solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{u}}_h \in \widehat{\boldsymbol{V}}_h$, and $p_h \in Q_h$. Moreover, there holds the a-priori estimate*

$$\|(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h)\|_{1,h} + \|p\|_{0,h} \leq C\|\boldsymbol{f}\|_{0,h},$$

with a constant C independent of the meshsize and the data.

Proof. Existence of a unique solution follows again from the discrete stability conditions and Brezzi's theorem [19, 21]. To show the a-priori bound, one utilizes

$$(\boldsymbol{f}, \boldsymbol{v}_h)_{\Omega^N} \leq \|\boldsymbol{f}\|_{\Omega^N}\|\boldsymbol{v}_h\|_{0,h} \leq C\|\boldsymbol{f}\|_{\Omega^N}\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h}.$$

The last inequality follows from Poincaré-Friedrichs inequalities for broken Sobolev spaces [18, eq. (1.3)], and the fact that $|\Gamma_{ij}| \geq h_E \simeq h_T$ for edges $E \subset \Gamma_{ij}$ and adjacent elements T; cf. Remark 3.1. □

4.4. **A-priori error estimates.** For obtaining a-priori error bounds, we require additional properties of the bilinear forms, and some interpolation error estimates.

Note that according to Lemma 3.4, there holds a Galerkin orthogonality, i.e.,

$$\begin{aligned} a_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\hat{\boldsymbol{u}}_h; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) + b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p-p_h) &= 0, \\ b_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\hat{\boldsymbol{u}}_h; q_h) &= 0, \end{aligned}$$

for all (discrete) test functions $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$, and $q_h \in Q_h$ as long as the solution $(\boldsymbol{u}, p)$ of the Stokes problem is sufficiently regular, i.e., such that a_h and b_h are well-defined.

As a next step, we state the boundedness of the bilinear forms with respect to a pair of stronger energy norms, namely

$$\begin{aligned} |\!|\!|(\boldsymbol{u}, \hat{\boldsymbol{u}})|\!|\!|_{1,h} &:= \left(\|(\boldsymbol{u}, \hat{\boldsymbol{u}})\|_{1,h}^2 + |\boldsymbol{\partial}_n \boldsymbol{u}|_{-1/2,h}^2\right)^{1/2}, \\ |\!|\!|p|\!|\!|_{0,h} &:= \left(\|p\|_{0,h}^2 + |p|_{-1/2,h}^2\right)^{1/2}. \end{aligned}$$

The following bounds follow with the same arguments as used for Proposition 4.5.

Proposition 4.12. *For all discrete functions $\boldsymbol{u}_h$, $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{u}}_h$, $\hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$, p_h, $q_h \in Q_h$, and all functions $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega^N)$, $p \in \boldsymbol{H}^1(\Omega^N)$, there holds*

$$a_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\hat{\boldsymbol{u}}_h; \boldsymbol{u}_h, \hat{\boldsymbol{v}}_h) \le C_a |\!|\!|(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\hat{\boldsymbol{u}}_h)|\!|\!|_{1,h} \|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h},$$

and

$$\begin{aligned} b_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\hat{\boldsymbol{u}}_h; q_h) &\le C_b |\!|\!|(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\hat{\boldsymbol{u}}_h)|\!|\!|_{1,h} \|q_h\|_{0,h}, \\ b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p-p_h) &\le C_b \|(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\hat{\boldsymbol{u}}_h)\|_{1,h} |\!|\!|p-p_h|\!|\!|_{0,h}, \end{aligned}$$

with mesh independent constants C_a and C_b as in Proposition 4.5.

The constants C_a, C_b in this theorem could be improved, but for ease of notation, we reused the ones from Proposition 4.5.

Remark 4.13. By Lemma 4.1, the norms $\|(\cdot,\cdot)\|_{1,h}$ and $|\!|\!|(\cdot,\cdot)|\!|\!|_{1,h}$, respectively $\|\cdot\|_{0,h}$ and $|\!|\!|\cdot|\!|\!|_{0,h}$ are equivalent on the finite dimensional spaces $\boldsymbol{V}_h$, $\widehat{\boldsymbol{V}}_h$, and Q_h, i.e.,

$$\begin{aligned} \|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h} \le |\!|\!|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)|\!|\!|_{1,h} &\le \sqrt{2}\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h}, \qquad \text{and} \\ \|q_h\|_{0,h} \le |\!|\!|q_h|\!|\!|_{0,h} &\le \sqrt{2}\|q_h\|_{0,h}, \end{aligned}$$

hold for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\hat{\boldsymbol{v}}_h \in \widehat{\boldsymbol{V}}_h$, and $q_h \in Q_h$. Hence, Proposition 4.5 in fact follows from Proposition 4.12 with slightly different constants.

As a final ingredient for the error analysis, we have to characterize the approximation properties of the finite element spaces: Let $\boldsymbol{\mathcal{I}}_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h$ be the standard nodal interpolation operator (defined subdomain-wise), and $\widehat{\boldsymbol{\Pi}}_h : \boldsymbol{H}_0^1(\Omega) \to \widehat{\boldsymbol{V}}_h$ and $\boldsymbol{\Pi}_h : L_0^2(\Omega) \to Q_h$ be the L^2 projection operators. The following interpolation error estimates follow with the usual scaling arguments; cf. e.g. [8, 24].

Lemma 4.14 (Approximation)**.** *For any $\boldsymbol{u} \in \boldsymbol{H}^2(\mathcal{T}_h) \times \boldsymbol{H}_0^1(\Omega)$ there holds*

$$|\!|\!|(\boldsymbol{u}-\boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \boldsymbol{u}-\widehat{\boldsymbol{\Pi}}_h\boldsymbol{u})|\!|\!|_{1,h} \le C\Big(\sum\nolimits_T h_T^2 \|\boldsymbol{u}\|_{2,T}^2\Big)^{1/2}, \tag{19}$$

and for every $p \in H^1(\mathcal{T}_h) \cap L_0^2(\Omega)$ there holds

$$|\!|\!|p - \Pi_h p|\!|\!|_{0,h} \le C\Big(\sum\nolimits_T h_T^2 \|p\|_{1,T}^2\Big)^{1/2}, \tag{20}$$

with constant C independent of the meshsize and the functions $\boldsymbol{u}$ and p.

Combining the consistency and the discrete stability of the hybrid mortar method with the boundedness and interpolation error estimates, we obtain the following a-priori error bound.

Theorem 4.15 (Energy-norm estimate)**.** *Let $(\boldsymbol{u}, p)$ denote the solution of the Stokes problem (2), and assume that $\boldsymbol{u} \in \boldsymbol{H}^2(\mathcal{T}_h)$ and $p \in H^1(\mathcal{T}_h)$. Moreover, let $(\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h, p_h)$ be the solution of Method 3.1. Then*

$$\|(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\hat{\boldsymbol{u}}_h)\|_{1,h} + \|p-p_h\|_{0,h} \le C\Big(\sum\nolimits_T h_T^2\left(\|\boldsymbol{u}\|_{2,T}^2 + \|p\|_{1,T}^2\right)\Big)^{1/2}$$

with a constant C independent of the meshsize and the functions.

Proof. By Theorem 4.10, Lemma 3.4, and Proposition 4.12, we obtain

$$\begin{aligned} c_{stab}(\|(\boldsymbol{u}_h - \boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \hat{\boldsymbol{u}}_h - \widehat{\boldsymbol{\Pi}}_h\boldsymbol{u})\|_{1,h} + \|p_h - \Pi_h p\|_{0,h}) \\ \le \big[a_h(\boldsymbol{u}_h - \boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \hat{\boldsymbol{u}}_h - \widehat{\boldsymbol{\Pi}}_h\boldsymbol{u}; \boldsymbol{v}_h, \hat{\boldsymbol{v}}_h) + b_h(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h; p - \Pi_h p) \\ + b_h(\boldsymbol{u}_h - \boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \hat{\boldsymbol{u}}_h - \widehat{\boldsymbol{\Pi}}_h\boldsymbol{u}; q_h)\big]/(\|(\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h)\|_{1,h} + \|q_h\|_{0,h}) \\ \le (C_a + C_b)\|\!|(\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}, \boldsymbol{u} - \widehat{\boldsymbol{\Pi}}_h\boldsymbol{u}\|\!|_{1,h} + C_b\|\!|p - \Pi_h p\|\!|_{0,h}, \end{aligned}$$

from which the estimate follows via the triangle inequality and the interpolation error estimates. □

Using the standard duality argument of Aubin-Nitsche, one can also obtain optimal error estimates with respect to the L^2-norm, provided that the continuous problem is sufficiently regular.

Corollary 4.16. *Assume that for any $\boldsymbol{f} \in L^2(\Omega)$, the solution $(\boldsymbol{u}, p)$ of problem (2) is in $\boldsymbol{H}^2(\mathcal{T}_h) \times H^1(\mathcal{T}_h)$ (cf. Remark 2.1). Then there exists a constant C independent of h, such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,h} \le Ch\left(\sum\nolimits_T h_T^2(\|\boldsymbol{u}\|_{2,\Omega^N}^2 + \|p\|_{1,\Omega^N}^2)\right)^{1/2},$$

where $h = \max_T h_T$ is the global meshsize.

Proof. The result follows from Theorem 4.15 and the usual duality argument. □

5. Remarks and generalizations

The aim of this section is to discuss the generalization of the previous results to three dimensional problems and discretizations using other finite element spaces. Let us shortly recall the basic assumptions needed in our analysis: We consider finite element spaces $\boldsymbol{V}_h$, Q_h made up of (conforming) piecewise polynomial functions on the subdomains, and a space $\widehat{\boldsymbol{V}}_h$ of piecewise polynomials on the interface.

Coercivity and boundedness. Propositions 4.4 and 4.5 rely on the assumption

$$(A1) \qquad |\boldsymbol{\partial}_n\boldsymbol{v}_h|_{-1/2,h} \le \tfrac{1}{2}\|\nabla\boldsymbol{v}_h\|_{\Omega^N} \qquad \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

which can always be fulfilled by an appropriate choice of the stabilization parameter function α, which is guided by the discrete trace inequality (8). Thus, if assumption $(A1)$ holds, coercivity and boundedness on the discrete level are proven as in Section 4.

Inf-sup stability. For the generalization of Theorem 4.9, we require additional conditions on the finite element spaces. Let

$$\boldsymbol{V}_h(\Omega_i) := \boldsymbol{V}_h|_{\Omega_i} \cap H_0^1(\Omega_i) \qquad \text{and} \qquad Q_h(\Omega_i) := Q_h|_{\Omega_i} \cap L_0^2(\Omega_i)$$

denote the restrictions of the global spaces to the subdomains. We assume that there exist constants β_i independent of h such that

$$(A2a) \qquad \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega_i)} \frac{b_i(\boldsymbol{v}_h, p_h)}{\|\boldsymbol{v}_h\|_{1,\Omega_i}} \geq \beta_i \|p_h\|_{0,\Omega_i} \qquad \text{for all} \quad p_h \in Q_h(\Omega_i).$$

Various inf-sup stable finite element pairs satisfying $(A2a)$ are known for different element types and degrees of approximation in two and three space dimensions; cf. e.g. [32, 21] or [14] for a recent overview. In addition, we require that the interface space $\widehat{\boldsymbol{V}}_h$ contains at least the constants for each subdomain interface Γ_{ij}, i.e.

$$(A2b) \qquad \mathbf{1}_{\Gamma_{ij}} \in \widehat{\boldsymbol{V}}_h, \qquad \text{for all } \Gamma_{ij} \in \Gamma^N.$$

This property is utilized for the second statement of Lemma 4.8. Apart from $(A2b)$, any (also non-polynomial) space can be used for approximation of the hybrid variables. To generalize also the first statement of Lemma 4.8 with mesh independent constants, we require certain approximation properties of the finite element spaces, namely the existence of an operator $\mathcal{C}_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h$ such that

$$(A2c) \qquad \|(\mathcal{C}_h \boldsymbol{v}, \widehat{\boldsymbol{\Pi}}_h \hat{\boldsymbol{v}}_h)\|_{1,h} \leq C'' \|\nabla \boldsymbol{v}\|_{\Omega^N} \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega).$$

As above, $\widehat{\boldsymbol{\Pi}}_h : L_0^2(\Gamma) \to \widehat{\boldsymbol{V}}_h$ denotes the L^2 projection. The construction of quasi-interpolation operators for the subdomains is well understood [23, 47], and due to the discontinuity of functions in $\boldsymbol{V}_h$ across the interfaces, condition $(A2c)$ essentially reduces to a simple assumption on the approximation properties of the spaces $\boldsymbol{V}_h$ and $\widehat{\boldsymbol{V}}_h$; see also the proof of Lemma 4.8. Under Assumptions $(A2a) - (A2c)$, the proof of Theorem 4.9 carries over verbatim to a wide class of discretizations.

Error estimates. The discrete stability and Galerkin orthogonality of the hybrid mortar method immediately imply the quasi-best approximation of the finite element solution, i.e., the a-priori estimates only depend on the approximation properties of the finite element spaces. The results for high-order approximations follows directly. Note that in general, the inf-sup stability constant will depend on the polynomial degree as well; cf. e.g. [46] and [28], where also a-posteriori estimates are derived.

Further remarks. At the end of this section, let us comment on some other generalizations: The incorporation of other boundary conditions is straightforward. Since we are already using variational arguments for dealing with the interface conditions, it seems natural to incorporate also the boundary conditions weakly, e.g., by Nitsche's method or hybrid variants. Results in this direction can be found for instance in [31].

In principle, also non-conforming discretizations, e.g. the non-conforming $\mathcal{P}_1 - \mathcal{P}_0$ element, could be chosen on the subdomains. In view of the variational coupling by a Nitsche-type technique, one may also think of utilizing discontinuous Galerkin discretizations already for the subdomain problems; cf. [30, 33] for approaches in this direction.

In principle, also a domain decomposition into single elements is possible, which results in a hybrid discontinuous Galerkin method for Stokes flow; cf. [41, 28] for related work. The conditions $(A1)$ and $(A2a) - (A2c)$ can then easily be verified, if the global mesh is generated from a conforming triangulation (eventually with hanging nodes).

The presented framework does also apply to the Oseen- and Navier-Stokes equations; we refer to [30, 33, 45, 49] for related work concerning discontinuous Galerkin methods. Also the variational coupling respectively unified treatment of Darcy and Stokes equations is possible, cf. [22]. Note that, the main ingredient for the analysis also of these problems is the discrete inf-sup condition for the incompressibility constraint, which has been established in this work.

6. Numerical Results

In this section, we present some results of numerical experiments for two model problems, namely a colliding flow and a backward facing step flow. We make a convergence study for hybrid mortar methods using different finite element discretizations, e.g., continuous $\mathcal{P}_k^c$ elements for the velocities on the subdomains, and continuous $\mathcal{P}_{k-1}^c$ or discontinuous $\mathcal{P}_{k-2}^d$ elements for the pressure. Moreover, we compare the results of the hybrid mortar method on non-matching triangulations with those obtained by continuous finite elements on conforming meshes. The numerical results have been computed with a finite element code based on the Dune framework [9]; in particular, we utilize the Grid-Glue module [6] for handling of the non-matching interfaces.

6.1. **Convergence studies.** As a first test problem, we consider a *colliding flow* on the square domain $\Omega = (-1,1)^2$. Boundary conditions are chosen, such that the exact solution is

$$\boldsymbol{u} = \left(20xy^3, 5x^4 - 5y^4\right)^\top, \quad p = 60x^2y - 20y^3.$$

The domain partition is given by $\Omega_1 = (-1,0)\times(-0.5,1)$, $\Omega_2 = (0,1)\times(-1,0.5)$, $\Omega_3 = (0,1)\times(0.5,1)$ and $\Omega_4 = (-1,0)\times(-1,-0.5)$; see also Figure 1 for a sketch of the domain partition and the initial mesh.

The hybrid mortar method is applied on a sequence of uniformly refined meshes. In Table 1, we list the L^2 errors for different inf-sup stable discretizations.

The numerical tests show the predicted convergence rates. Since the exact solution is a polynomial of order 4, and the method is consistent, we obtain the exact solution when using a method of order $k = 4$.

6.2. **Backward facing step.** As a second example, we consider the *backward facing step flow* problem on the geometry depicted in Figure 2. For the hybrid mortar method, the domain is partitioned into three subdomains $\Omega_1 = (-2,-0.5)\times(0.5,1)$, $\Omega_3 = (1,10)\times(0,1)$ and $\Omega_2 = \Omega\backslash(\Omega_1\cup\Omega_2)$. At the in- and outflow boundaries, we impose parabolic velocity profiles

$$\boldsymbol{u}(-2,y) = (\,8\,(1-y)\,(y-0.5)\,,0\,)^\top \quad \text{and} \quad \boldsymbol{u}(10,y) = (\,y(1-y),0\,)^\top,$$

and on the rest of the boundary we apply a no-slip condition $\boldsymbol{u} \equiv 0$. The compatibility condition $\int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n}\,dx = 0$, which stems from the incompressibility of the fluid, is satisfied by our choice of boundary data.

For discretization of the backward-facing-step problem, we use the second order $(\mathcal{P}_2^c - \mathcal{P}_1^c)$ Taylor-Hood element. The solutions obtained with fourth order Taylor-Hood element are used for estimating the discretization error. Again, we run a series of computations on uniformly refined meshes.

In our tests, we also compare the hybrid mortar methods on non-matching grids to the corresponding standard methods on a conforming mesh. The two initial meshes

h	$\alpha = 2k^2h^{-1}$			
$\mathcal{P}_2^c - \mathcal{P}_0^d$	L^2	rate	energy	rate
1.0	$7.64 \cdot 10^{0}$	–	$1.33 \cdot 10^{1}$	–
0.5	$1.93 \cdot 10^{0}$	1.97	$6.92 \cdot 10^{0}$	0.95
0.25	$4.85 \cdot 10^{-1}$	1.99	$3.50 \cdot 10^{0}$	0.98
0.125	$1.21 \cdot 10^{-1}$	1.99	$1.76 \cdot 10^{0}$	0.99
$\mathcal{P}_2^c - \mathcal{P}_1^c$	L^2	rate	energy	rate
1.0	$1.41 \cdot 10^{0}$	–	$3.32 \cdot 10^{0}$	–
0.5	$1.71 \cdot 10^{-1}$	3.05	$7.73 \cdot 10^{-1}$	2.10
0.25	$2.11 \cdot 10^{-2}$	3.01	$1.84 \cdot 10^{-1}$	2.06
0.125	$2.63 \cdot 10^{-3}$	3.00	$4.50 \cdot 10^{-2}$	2.03
$\mathcal{P}_3^c - \mathcal{P}_2^c$	L^2	rate	energy	rate
1.0	$9.90 \cdot 10^{-2}$	–	$2.25 \cdot 10^{-1}$	–
0.5	$5.94 \cdot 10^{-3}$	4.05	$2.72 \cdot 10^{-2}$	3.05
0.25	$3.73 \cdot 10^{-4}$	3.99	$3.35 \cdot 10^{-3}$	3.02
0.125	$2.41 \cdot 10^{-5}$	3.94	$4.20 \cdot 10^{-4}$	2.99

TABLE 1. L^2 and H^1 (energy-norm) errors for the colliding flow problem for different inf-sup stable finite element approximations on a sequence of uniformly refined meshes. The interface space is discretized by piecewise $\mathcal{P}_k$ elements, where k denotes the order of the velocity approximation. The stabilization parameter was chosen by $\alpha = 2k^2h^{-1}$; see also [35, 43].

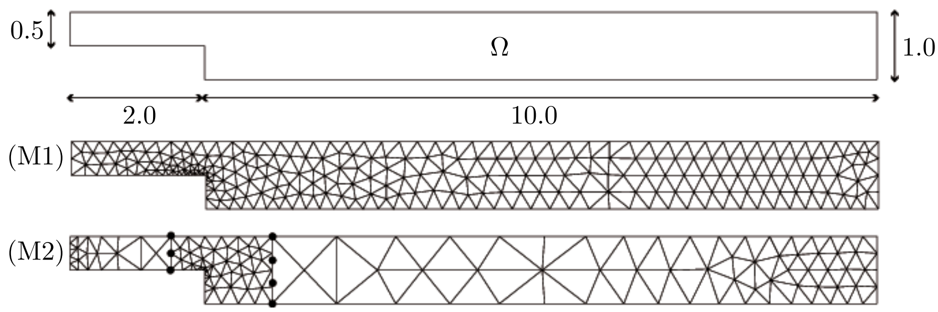

FIGURE 2. Geometry and initial triangulations of the backward facing step domain. Bullets denote interface nodes. (M1) consists of 602 triangular elements and (M2) has 307 triangular elements and 5 interface elements.

(M1) and (M2) used for our computations are shown in Figure 2. They were chosen such that the errors of the two finite element methods are of comparable size on the coarsest level. The initial meshes were also slightly refined towards the inward corner, where the solution lacks full regularity.

In Table 6.2, we list the observed convergence rates for the mortar finite element method on non-conforming meshes based on (M1), and for the standard finite element method on conforming meshes generated from (M2).

	Conforming (M1)				Hybrid Mortar (M2)			
h	L^2 error	rate	energy err.	rate	L^2 error	rate	energy err.	rate
1.0	$1.54 \cdot 10^{-1}$	–	$2.80 \cdot 10^{-1}$	–	$1.45 \cdot 10^{-1}$	–	$2.87 \cdot 10^{-1}$	–
0.5	$4.88 \cdot 10^{-2}$	1.65	$1.77 \cdot 10^{-1}$	0.66	$4.36 \cdot 10^{-2}$	1.73	$1.77 \cdot 10^{-1}$	0.69
0.25	$1.64 \cdot 10^{-2}$	1.56	$1.19 \cdot 10^{-1}$	0.57	$1.44 \cdot 10^{-2}$	1.59	$1.07 \cdot 10^{-1}$	0.72
0.125	$5.70 \cdot 10^{-3}$	1.53	$8.17 \cdot 10^{-2}$	0.54	$4.91 \cdot 10^{-3}$	1.55	$7.26 \cdot 10^{-2}$	0.56

Table 2. L^2 errors for the backward facing step problem obtained with second order ($\mathcal{P}_2^c - \mathcal{P}_1^c$) Taylor-Hood elements. The results on the left are obtained with the standard finite element method on conforming meshes generated by uniform refinement of (M1). The meshes for the hybrid mortar method are based on the initial triangulation (M2).

The results obtained with the both methods are very similar, i.e., the difference in discretization errors remains comparable in all computations. As expected, the convergence rates for the uniform refinement study are limited by the corner singularity of the solution.

7. Conclusions

In this article, we proposed and analyzed a hybrid mortar method for Stokes interface problems on non-matching grids. Discrete stability could be shown for a large class of inf-sup stable finite elements under mild assumptions on the approximation spaces. Optimal energy- and L^2-norm error estimates could be derived, and the theoretical results were illustrated by numerical experiments.

The conditions needed for our stability analysis are rather weak, i.e., in principle it suffices that the interface space contains at least one degree of freedom for every subdomain interface. This condition is automatically satisfied, if the interface mesh resolves the partition of the skeleton, or when the interface mesh is sufficiently fine. In contrast to other approaches, the subdomain meshes can be chosen completely independent from each other.

Our analysis also covers, as a limiting case, a class of hybrid discontinuous Galerkin methods for Stokes flow, which can be interpreted as domain partition into single elements. The basic proof of the inf-sup stability conditions allows the generalization of the framework to Oseen and Navier-Stokes equations. This and the use of the hybrid method in domain decomposition algorithms are subjects of ongoing research.

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